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REPRESENTING $PSl_2(p)$ ON A RIEMANN SURFACE OF LEAST GENUS

by Henry GLOVER and Denis SJERVE [1])

§ 1. INTRODUCTION

Given any finite group G there exists a closed Riemann surface S and an effective action $G \times S \to S$ by conformal automorphisms (here conformal means analytic). Therefore it makes sense to ask what is the least genus of such surfaces S. Recall that when the answer is that the genus equals zero (i.e. G acts on the two sphere) then G is from the list $\mathbf{Z}/n$, D_n, A_4, S_4 or A_5. The purpose of this paper is to determine this minimum genus for the simple groups $PSl_2(p)$, where $p \geqslant 5$ is a prime. Since given any finite group G and Riemann surface T there exists a regular branched covering $p: S \to T$ such that i) G is the group of branched covering transformations of p (i.e. $T = S/G$) and ii) G is the full group of automorphisms of S [Gr], it seems most interesting to realize G as the full group of automorphisms of a Riemann surface of least genus. In a sequel to this paper [GS] we will prove that this always happens when $p \not\equiv \pm 1 \bmod 8$ or mod 5 but *may* fail for these congruence equalities. When it does fail $PSl_2(p)$ will have index two in the full group of automorphisms. In addition, a particularly simple situation occurs when $p: S \to S/G$ has exactly three branch points. Our results always give this for $PSl_2(p)$. We conjecture analogus results for every finite simple group and we seek to relate these ideas to "moonshine" for simple groups [FLM]. In order to state our results we need some notation:

(1) $PSl_2(p^k)$ is the projective special linear group of 2×2 matrices over the Galois field $GF(p^k)$.

(2) $\Gamma = PSl_2(\mathbf{Z})$ is the classical modular group. Geometrically Γ is just the group of integral linear fractional transformations of the upper half plane H, that is transformations of the form $z \to \dfrac{az + b}{cz + d}$, where a, b, c, d

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are integers so that $ad - bc = 1$. Algebraically Γ is the unimodular group $Sl_2(\mathbf{Z})$ modulo its center $= \{\pm I\}$.

A result of Newman [N] is that mod p reduction of entries gives an epimorphism $\Gamma \twoheadrightarrow PSl_2(p)$, and therefore an exact sequence $1 \to \Delta \to \Gamma \to PSl_2(p) \to 1$. Now Δ is a Fuchsian group and therefore $PSl_2(p)$ is acting conformally on the open Riemann surface H/Δ. By adding parabolic points we obtain a closed Riemann surface $\overline{H/\Delta}$ and a conformal action on $\overline{H/\Delta}$ by extension. According to [G] the genus of $\overline{H/\Delta}$ is

$$1 + \frac{|PSl_2(p)|}{2}\left(\frac{1}{6} - \frac{1}{p}\right) = 1 + \frac{p(p^2-1)}{4}\left(\frac{1}{6} - \frac{1}{p}\right)$$

where $|PSl_2(p)| = \dfrac{p(p^2-1)}{2}$ is the order of $PSl_2(p)$.

Definition. For any finite group G we let *genus* (G) denote the least genus of all Riemann surfaces S for which there exists an effective conformal action $G \times S \to S$. We note that genus (G) has also been called the symmetric genus of G in the literature.

Thus we certainly have genus $(PSl_2(p)) \leqslant 1 + \dfrac{p(p^2-1)}{4}\left(\dfrac{1}{6} - \dfrac{1}{p}\right)$. Putting $p = 5$ then gives genus $(PSl_2(5)) = 0$, and therefore we will tacitly assume in all that follows that $p \geqslant 7$.

For $p = 7, 11$ we get the inequalities genus $(PSl_2(7)) \leqslant 3$ and genus $(PSl_2(11)) \leqslant 26$. It will turn out that these inequalities are equalities (see the corollary of the introduction). The action of $PSl_2(7)$ on a surface of genus 3 is the action of the simple group of order 168 considered by Klein.

This inequality strongly suggests that genus $(PSl_2(p))$ can be calculated by realizing $PSl_2(p)$ as an epimorphic image of Γ, or some other Fuchsian group, and then minimizing over all such epimorphisms. For example Γ has the presentation:

$$\Gamma = \{S, T \mid S^2 = (ST)^3 = 1\},$$

$$\text{where} \quad S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Reducing coefficients mod p leads to a presentation of $PSl_2(p)$, namely

$$PSl_2(p) = \{A, B, C \mid A^2 = B^3 = C^p = ABC = 1, ETC\}$$

where we have made the substitutions $A = S$, $B = ST$ and $C = T^{-1}$. We have written the presentation in this manner so that it becomes clear that $PSl_2(p)$ is an epimorphic image of the triangle group

$$T(2, 3, p) = \{A, B, C \mid A^2 = B^3 = C^p = ABC = 1\}.$$

Recall that if r, s, t are integers $\geqslant 2$ then $T(r, s, t)$ is the group of orientation preserving symmetries of the appropriate plane generated by rotations of $2\pi/r$, $2\pi/s$ and $2\pi/t$, respectively, about the vertices of a triangle having angles π/r, π/s and π/t respectively. The plane is spherical if $1/r + 1/s + 1/t > 1$, euclidean if $1/r + 1/s + 1/t = 1$, and hyperbolic if $1/r + 1/s + 1/t < 1$. See Magnus [M] for more details.

Using the above presentation of $PSl_2(p)$ leads to an exact sequence $1 \to \Delta \to T(2, 3, p) \to PSl_2(p) \to 1$ and an effective conformal action of $PSl_2(p)$ on the closed Riemann surface H/Δ. Again we have

$$\text{genus } (H/\Delta) = 1 + \frac{p(p^2-1)}{4}\left(\frac{1}{6} - \frac{1}{p}\right)$$

so there is no improvement. But now the idea is clear: find all triples (r, s, t) for which there is an exact sequence $1 \to \Delta \to T(r, s, t) \to PSl_2(p) \to 1$, compute the genus of H/Δ for any such extension, and then minimize over all possible triples. It turns out that this procedure gives genus $(PSl_2(p))$ because more branch points always gives a higher genus.

If $p \geqslant 13$ we make the definition $d = \min\{e \mid e \geqslant 7$ and either $e \mid \frac{p-1}{2}$ or $e \mid \frac{p+1}{2}\}$. Then our results are:

THEOREM I. *Assume* $p \geqslant 13$. *Then there exists a short exact sequence* $1 \to \Delta \to T(2, 3, d) \to PSl_2(p) \to 1$ *and the genus of* H/Δ *is*

$$1 + \frac{p(p^2-1)}{4}\left(\frac{1}{6} - \frac{1}{d}\right).$$

THEOREM II.

(a) *If* $p \equiv \pm 1\,(5)$ *then there exists a short exact sequence* $1 \to \Delta \to T(2, 5, 5) \to PSl_2(p) \to 1$ *and the genus of* H/Δ *is* $1 + \dfrac{p(p^2-1)}{40}$.

(b) *If* $p \equiv \pm 1\,(8)$ *then there exists a short exact sequence* $1 \to \Delta \to T(3, 3, 4) \to PSl_2(p) \to 1$ *and the genus of* H/Δ *is* $1 + \dfrac{p(p^2-1)}{48}$.

(c) *If* $p \equiv \pm 1\,(5)$ *and* $p \equiv \pm 1\,(8)$ *then there exists a short exact sequence* $1 \to \Delta \to T(2,4,5) \to PSl_2(p) \to 1$ *and the genus of* H/Δ *is* $1 + \dfrac{p(p^2-1)}{80}$.

Then we will prove that genus $(PSl_2(p))$ is obtained by minimizing over all the possibilities above.

The result of this minimization is

COROLLARY. *The genus of* $PSl_2(p)$ *is given as follows:*

(a) $g = 1 + \dfrac{p(p^2-1)}{4}\left(\dfrac{1}{6} - \dfrac{1}{p}\right)$ *if* $p = 5, 7, 11$,

(b) $g = 1 + \dfrac{p(p^2-1)}{40}$ *if* $p \geqslant 13$, $p \equiv \pm 1\,(5)$, $p \not\equiv \pm 1\,(8)$ *and* $d \geqslant 15$,

(c) $g = 1 + \dfrac{p(p^2-1)}{48}$ *if* $p \geqslant 13$, $p \not\equiv \pm 1\,(5)$, $p \equiv \pm 1\,(8)$ *and* $d \geqslant 12$,

(d) $g = 1 + \dfrac{p(p^2-1)}{80}$ *if* $p \geqslant 13$, $p \equiv \pm 1\,(5)$, $p \equiv \pm 1\,(8)$ *and* $d \geqslant 9$,

(e) $g = 1 + \dfrac{p(p^2-1)}{4}\left(\dfrac{1}{6} - \dfrac{1}{d}\right)$ *in all other cases.*

In fact the least genus g always comes from the branched covering space action on the Riemann surface $S = H/\Delta$ associated to some extension $1 \to \Delta \to T(r,s,t) \to PSl_2(p) \to 1$, where

$$(r,s,t) = \begin{cases} (2,3,p) & \text{if } p = 5, 7, 11, \\ (2,5,5) & \text{if } p \geqslant 13,\ p \equiv \pm 1\,(5),\ p \not\equiv \pm 1\,(8) \text{ and } d \geqslant 15, \\ (3,3,4) & \text{if } p \geqslant 13,\ p \not\equiv \pm 1\,(5),\ p \equiv \pm 1\,(8) \text{ and } d \geqslant 12, \\ (2,4,5) & \text{if } p \geqslant 13,\ p \equiv \pm 1\,(5),\ p \equiv \pm 1\,(8) \text{ and } d \geqslant 9, \\ (2,3,d) & \text{in all other cases.} \end{cases}$$

It turns out that other triples (r,s,t) are not relevant for the determination of the minimal genus.

In most cases the answer is $(r,s,t) = (2,3,d)$. For $p \leqslant 617$ the triple $(2,5,5)$ occurs once exactly, namely for $p = 509$, $(3,3,4)$ occurs exactly three times, namely for $p = 103$, 137 and 569 and (2, 4, 5) occurs exactly six times, for $p = 199$, 239, 359, 439, 521 and 599.

If $S = H/\Delta$ is the surface of minimal genus for $PSl_2(p)$ coming from one of the extensions above then the orbit manifold $S/PSl_2(p)$ is the 2-sphere S^2 and the quotient map $S \to S^2$ is a branched covering with exactly 3 branch points. One of the most important steps in the proof of the main result of this paper is the converse, namely if S is a Riemann surface of least genus for the group $G = PSl_2(p)$ then $S/G = S^2$ and $S \to S^2$ is a branched covering with exactly 3 branch points (see section 3). Note that a related notion of genus, "the Cayley genus of a group" has been studied by others, among them Tucker [T]. Earlier results can be found in Hurwitz [H] and Burnside [B].

The remainder of this paper is organized as follows. In section 2 we describe various ways of generating $PSl_2(p)$ and then prove theorems I and II. Section 3 proves that if S is a Riemann surface of least genus for $PSl_2(p)$ then $S/PSl_2(p)$ is a 2-sphere S^2 and the branched covering $S \to S^2$ has exactly 3 branch points. The calculation of genus $(PSl_2(p))$ then follows from the results of section 2.

Finally we would like to thank Bomshik Chang for help with the group theory of $PSl_2(p)$. The first author would like to thank the University of British Columbia for its hospitality to him during the time this research was done.

§ 2. Generating Triples for $PSl_2(p)$

Our goal in this section is to find triples (r, s, t) for which there are epimorphisms $T(r, s, t) \twoheadrightarrow PSl_2(p)$. In other words, given integers $r, s, t \geqslant 2$ are there matrices $A, B, C \in PSl_2(p)$ so that A, B, C generate $PSl_2(p)$ and $A^r = B^s = C^t = ABC = 1$? Throughout this section a standard reference for the group theory is Suzuki [S].

The spherical triangle groups are given in the following table

Table I

triple	*triangle group*	*order*
$(2, 2, n)$	dihedral	$2n$
$(2, 3, 3)$	tetrahedral (A_4)	12
$(2, 3, 4)$	octahedral (S_4)	24
$(2, 3, 5)$	icosahedral (A_5)	60

Now the group $PSl_2(p)$ has an element of order p since its order is $|PSl_2(p)| = \frac{p(p^2-1)}{2}$. It therefore follows that $PSl_2(p)$ is not the image of any spherical triangle group since $PSl_2(p)$ can not be the image of any dihedral group and we are assuming $p \geqslant 7$. The following lemma then implies that $PSl_2(p)$ can only be the image of hyperbolic triangle groups.

(2.1). LEMMA. *$PSl_2(p)$ is not the image of any euclidean triangle group.*

Proof. Suppose T is one of the euclidean triangle groups, namely one of $T(3, 3, 3)$, $T(2, 4, 4)$, $T(2, 3, 6)$, and there exists an epimorphism $T \twoheadrightarrow PSl_2(p)$. Since T has $\mathbf{Z} \oplus \mathbf{Z}$ as a normal subgroup of index $\leqslant 6$ it follows that $PSl_2(p)$ has an abelian normal subgroup of index $\leqslant 6$. But this is clearly not possible. Q.e.d.

In order to decide when a triple of matrices $A, B, C \in PSl_2(p)$ generates the entire group we need detailed knowledge of the maximal subgroups. The following theorem can be found in Suzuki [S].

(2.2). THEOREM. *The maximal proper subgroups of $PSl_2(p)$ are:*

(a) *dihedral of order* $p-1$ *or* $p+1$.

(b) *solvable of order* $\frac{p(p-1)}{2}$.

(c) A_4 *if* $p \equiv 3, 13, 27, 37 \bmod 40$.

(d) S_4 *if* $p \equiv \pm 1 \bmod 8$.

(e) A_5 *if* $p \equiv \pm 1 \bmod 5$.

The dihedral group of order $p-1$ can be chosen to be

$$D = \langle R, S \rangle = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}, \begin{bmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{bmatrix} \mid \alpha \in \mathbf{Z}_p^* \right\}, \quad \text{where}$$

$$R = \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}, \; S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and } x \text{ is a primitive root mod } p.$$

To realize the dihedral subgroup of order $p+1$ we need another description of $PSl_2(p)$. The mapping

$$GF(p^2) \to GF(p^2),\ x \to x^p$$

is an automorphism of order 2. For convenience we put $\bar{x} = x^p$. Then $PSl_2(p) \cong PSU_2(p)$, where $PSU_2(p)$ is the projective special unitary group

$$PSU_2(p) = \left\{\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \mid a, b \in GF(p^2), a\,\bar{a} + b\,\bar{b} = 1\right\}$$

Now consider the matrix $U = \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix}$, where $\omega \in GF(p^2)$ is chosen so that $\omega^{(p+1)/2} = -1$ and $\omega^k \neq \pm 1$ for $1 \leqslant k < \dfrac{p+1}{2}$. Then the order of U as an element of $PSU_2(p)$ is $\dfrac{p+1}{2}$ and the dihedral group of order $\dfrac{p+1}{2}$ can be taken to be

$$D = \langle U, S \rangle = \left\{\begin{bmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{bmatrix}, \begin{bmatrix} 0 & \alpha \\ -\bar{\alpha} & 0 \end{bmatrix} \mid \alpha \in GF(p^2)^*, \alpha^{p+1} = 1\right\}.$$

Finally the maximal solvable subgroup of order $\dfrac{p(p-1)}{2}$ can be chosen to be the subgroup of upper triangular matrices

$$H = \left\{\begin{bmatrix} x & \lambda \\ 0 & x^{-1} \end{bmatrix} \mid x \in \mathbf{Z}_p^*, \lambda \in \mathbf{Z}_p\right\}.$$

Thus there is a split extension of the form

$$1 \to \mathbf{Z}_p \to H \xrightarrow{\theta} \mathbf{Z}_{(p-1)/2} \to 1, \theta\colon \begin{bmatrix} x & \lambda \\ 0 & x^{-1} \end{bmatrix} \to \pm x.$$

The kernel is generated by $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and the splitting is induced by the matrix $\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}$, where x is a primitive root mod p.

The other maximal subgroups will not play much of a role in what follows. Notice that an immediate consequence of (2.2) is

(2.3). LEMMA.

(a) *The order of an element of* $PSl_2(p)$ *is one of the following: a divisor of either* $\dfrac{p-1}{2}$ *or* $\dfrac{p+1}{2}$; p; 2, 3, 4 *or* 5.

(b) *If* d *is a divisor of either* $\frac{p-1}{2}$ *or* $\frac{p+1}{2}$ *then there is an element of* $PSl_2(p)$ *having order* d.

The order of an element $A \in PSl_2(p)$ can be determined from its trace. In particular we have:

(2.4) LEMMA. *Let* $A \in PSl_2(p)$ *and* $\chi = \pm$ trace A. *Then the order of* A *is* $2, 3, 4,$ *or* 5 *respectively if, and only if,* $\chi \equiv 0\,(p)$, $\chi \equiv \pm 1\,(p)$, $\chi^2 \equiv 2\,(p)$ *or* $\chi^2 \pm \chi - 1 \equiv 0\,(p)$ *respectively.*

Definition. We say that a triple of elements (A, B, C) from $PSl_2(p)$ is an (r, s, t) triple if (a) order $A = r$, order $B = s$, order $C = t$; and (b) $ABC = 1$.

In order to construct $(2, 3, d)$ triples for $d \mid \frac{p-1}{2}$ let A, B, C be the matrices

$$(2.5)\ A = \begin{bmatrix} 0 & -x \\ x^{-1} & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix},\ C = (AB)^{-1} = \begin{bmatrix} x^{-1} & x \\ 0 & x \end{bmatrix}$$

where $x \in \mathbf{Z}_p^*$. Then order $A = 2$, order $B = 3$ and

$$C^k = \begin{bmatrix} x^{-k} & x(x^{k-1} + x^{k-3} + \dots + x^{-(k-1)}) \\ 0 & x^k \end{bmatrix}$$

If $x = \pm 1$ then $C = T$ and order $T = p$. In general the order of C is given by the following lemma whose proof is elementary and hence omitted.

(2.6). LEMMA. *Assume* $x \neq \pm 1$. *Then the order of* C *in* $PSl_2(p)$ *is the least positive integer* k *so that either* $x^k = 1$ *or* $x^k = -1$.

Given $x \in \mathbf{Z}_p^*$, $x \neq \pm 1$, let k be the least positive integer so that $x^k = \pm 1$. Since we always have $x^{(p-1)/2} = \pm 1$ it follows that $1 < k \leqslant \frac{p-1}{2}$. Also $x^{2k} = 1$ and therefore $k \mid \frac{p-1}{2}$. Conversely, given any divisor d of $\frac{p-1}{2}$ there exists $x \in \mathbf{Z}_p^*$ so that d is the least positive integer k satisfying $x^k = \pm 1$.

(2.7). COROLLARY. *Suppose* $d > 1$ *is a divisor of* $\frac{p-1}{2}$. *Then there exist* $(2, 3, d)$ *triples* (A, B, C) *in* $PSl_2(p)$.

Next we determine when there are $(2, 3, d)$ triples for divisors of $\frac{p+1}{2}$. Suppose $x \in GF(p^2)^*$ is such that $x^{p+1} = 1$. Then consider the triple of matrices (A, B, C) in $PSU_2(p)$:

$$(2.8).\qquad A = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \quad B = \begin{bmatrix} \bar{x}\,\bar{a} & -xb \\ \bar{x}\,\bar{b} & xa \end{bmatrix}, \quad C = \begin{bmatrix} x & 0 \\ 0 & \bar{x} \end{bmatrix}$$

where $a, b \in GF(p^2)$ satisfy $a\,\bar{a} + b\,\bar{b} = 1$.

It is easy to check that $ABC = 1$.

(2.9). LEMMA. *Let* $d > 2$ *be any divisor of* $\frac{p+1}{2}$. *Then there are* $(2, 3, d)$ *triples in* $PSl_2(p)$.

Proof. Let $x \in GF(p^2)^*$ be any element so that d is the least positive integer satisfying $x^d = \pm 1$. Then the matrix C in (2.8) has order d. Next we choose $a \in GF(p^2)^*$ so that $a(x - x^{-1}) = 1$. Since

$$GF(p) = \{b\,\bar{b} \mid b \in GF(p^2)\}$$

it follows that there exists $b \in GF(p^2)$ such that $a\,\bar{a} + b\,\bar{b} = 1$.

We now prove that the matrices A, B of (2.8) have orders 2, 3 respectively, that is we will show that $a + \bar{a} = 0$ and $ax + \bar{a}\,\bar{x} = \pm 1$. Since $x^{p+1} = 1$ we have

$$1 = a^p(x - x^{-1})^p = a^p(x^p - x^{-p}) = a^p(x^{-1} - x)\,.$$

This together with $1 = a(x - x^{-1})$ implies that $a^p = -a$, i.e., $a + \bar{a} = 0$. Finally

$$ax + \bar{a}\,\bar{x} = ax + a^p\,x^p = ax - ax^{-1} = a(x - x^{-1}) = 1\,. \qquad \text{Q.e.d.}$$

The next theorem proves one half of theorem I of the introduction.

(2.10). THEOREM. *Suppose* d *is a divisor of either* $\frac{p-1}{2}$ *or* $\frac{p+1}{2}$ *and suppose* $d > 6$. *Then there is a* $(2, 3, d)$ *triple* (A, B, C) *so that the group generated by* A, B, C *is* $PSl_2(p)$.

Proof. Let (A, B, C) be any $(2, 3, d)$ triple and set $G = \langle A, B, C \rangle$ = the subgroup generated by A, B, C. Since G has elements of order $d > 6$ it follows that G can not be a subgroup of A_4, S_4, A_5. Therefore, if $G \neq PSl_2(p)$, it follows that either $G \subseteq D$ or $G \subseteq H$, where D is a maximal dihedral subgroup and H is a maximal solvable subgroup (see (2.2)).

First we assume that $G \subseteq D$. Since B, ABA both have order 3 they must commute, i.e., $(AB)^2 = (BA)^2$. But then we have

$$(AB)^6 = (AB)^2AB(AB)^2AB = (BA\,BA)AB\,(BA\,BA)AB = BAB^2BAB^2 = 1$$

contradicting our hypothesis that $C = (AB)^{-1}$ has order $d > 6$.

Next assume that $G \subseteq H$. Since there is an extension

$$1 \to \mathbf{Z}_p \to H \xrightarrow{\theta} \mathbf{Z}_{(p-1)/2} \to 1$$

we see that $(AB)^6 \in \mathbf{Z}_p$ since A has order 2, B has order 3, and $\theta(A)$ and $\theta(B)$ commute. If $d \mid \dfrac{p-1}{2}$ then

$$1 = (AB)^{6p} = (AB)^{6\left(\frac{p-1}{2} + \frac{p-1}{2} + 1\right)} = (AB)^6 \quad \text{since} \quad (AB)^{\frac{p-1}{2}} = 1\,.$$

This contradicts the fact that AB has order $d > 6$. The argument for divisors of $\dfrac{p+1}{2}$ is similar. Q.e.d.

Summarizing we now know that $PSl_2(p)$ is generated by a $(2, 3, p)$ triple and also by any $(2, 3, d)$ triple, where $d > 6$ and d is a divisor of either $\dfrac{p-1}{2}$ or $\dfrac{p+1}{2}$. As far as the problem of minimum genus is concerned it turns out that in addition we only need determine those primes p for which $PSl_2(p)$ is generated by a triple of the form $(3, 3, 4)$, $(2, 5, 5)$, $(2, 4, 5)$.

According to (2.4) a matrix $C \in PSl_2(p)$ has order 4, respectively order 5, if, and only if, $\chi^2 \equiv 2\,(p)$, respectively $\chi^2 \pm \chi - 1 \equiv 0\,(p)$, where $\chi = \operatorname{trace} C$. But these equations are solvable over $\mathbf{Z}_p$ if, and only if, $p \equiv \pm 1\,(8)$, respectively $p \equiv \pm 1\,(5)$. Since every element of $\mathbf{Z}_p$ can arise as the trace of some matrix we have $PSl_2(p)$ has elements of order 4, respectively order 5, if, and only if, $p \equiv \pm 1\,(8)$, respectively $p \equiv \pm 1\,(5)$.

To construct $(3, 3, 4)$ triples consider matrices

(2.11). $$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ c & -a+1 \end{bmatrix},$$

$$C = (AB)^{-1} = \begin{bmatrix} 1-a-b & a-1 \\ a-c & c \end{bmatrix}$$

where $-a^2 + a - bc \equiv 1\,(p)$.

A and B both have order 3 and C will have order 4 if, and only if, $(1-a-b+c)^2 \equiv 2\,(p)$. Therefore we need to find a, b, c satisfying

(2.12). $$-a^2 + a - bc \equiv 1\,(p) \quad \text{and} \quad (1-a-b+c)^2 \equiv 2\,(p).$$

Assume $p \equiv \pm 1\,(8)$ so that there is $\alpha \in \mathbf{Z}_p$ with $\alpha^2 \equiv 2\,(p)$. Then (2.12) is equivalent to

$$1 - a - b + c \equiv \alpha \quad \text{and} \quad a^2 - a + bc + 1 \equiv 0$$

which in turn is equivalent to finding b, c so that

(2.13). $$-3 - 4bc \text{ is a quadratic residue mod } p \quad \text{and}$$
$$\frac{1 \pm \sqrt{-3 - 4bc}}{2} \equiv 1 - b + c - \alpha\,.$$

But this is the same as finding b, c so that

(2.14). $$-3 - 4bc \equiv \big(1 + 2(-b + c - \alpha)\big)^2\,.$$

Now solving for c we see that there is a solution, if, and only if, $-3b^2 + (2-4\alpha)b - 3$ is a quadratic residue for some choice of b. But quadratic polynomials always assume at least one quadratic residue and therefore it is possible to satisfy (2.12).

Thus we have proved the following theorem.

(2.15). THEOREM. *Suppose* $p \equiv \pm 1\,(8)$. *Then there are* $(3, 3, 4)$ *triples in* $PSl_2(p)$, *one such being given by* (2.11), *where* a, b, c *are chosen to satisfy*

$$-a^2 + a - bc \equiv 1\,(p) \quad \textit{and} \quad (1-a-b+c)^2 \equiv 2\,(p)\,.$$

We still must prove that $PSl_2(p)$ can be generated by a $(3, 3, 4)$ triple if $p \equiv \pm 1\,(8)$.

(2.16). THEOREM. *Suppose* $p \equiv \pm 1\,(8)$. *Then there are* $(3, 3, 4)$ *triples in* $PSl_2(p)$ *and any such triple will generate* $PSl_2(p)$.

Proof. Let (A, B, C) be any $(3, 3, 4)$ triple, which exists by (2.15), and let $G = \langle A, B, C \rangle$. We use (2.2) to prove that $G = PSl_2(p)$. First note that none of A_4, S_4, A_5 contain $(3, 3, 4)$ triples. Secondly suppose that $G \subseteq D$, where D is a dihedral group. Since A, B are elements, of odd order (in a dihedral group) they commute and consequently AB will not have order 4.

Finally, suppose $G \subset H$, where H is a maximal solvable subgroup of $PSl_2(p)$. From the existence of the extension $1 \to \mathbf{Z}_p \to H \xrightarrow{\theta} \mathbf{Z}_{(p-1)/2} \to 1$ we see that $AB \in \mathbf{Z}_p$ since $\theta(AB)^4 = 1$ and $\theta(AB)^3 = 1$. But this is impossible since the order of AB is 4. Q.e.d.

To construct (2, 5, 5) or (2, 4, 5) triples in the case $p \equiv 1\,(5)$ consider the matrices

$$(2.17).\quad A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix},\ B = \begin{bmatrix} -ax^{-1} & -bx \\ -cx^{-1} & ax \end{bmatrix},\ C = \begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}$$

where $a, b, c, x \in GF(p)$ are chosen so that

$$-a^2 - bc = 1,\ x^5 = 1,\ x \neq \pm 1\,.$$

If we also have $p \equiv \pm 1\,(8)$ then we can choose a so that $a^2(x-x^{-1})^2 = 2$, and therefore (A, B, C) will be a (2, 4, 5) triple. On the other hand choosing a so that $\alpha = a(x-x^{-1})$ is a solution of $u^2 \pm u - 1 = 0$ will guarantee that (A, B, C) is a (2, 5, 5) triple.

In the case $p \equiv -1\,(5)$ we think of $PSl_2(p)$ as the projective special unitary group $PSU_2(p)$. Thus we have the matrices

$$(2.18).\quad A = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix},\ B = \begin{bmatrix} \bar{a}\,\bar{x} & -bx \\ \bar{b}\,\bar{x} & ax \end{bmatrix},\ C = \begin{bmatrix} x & 0 \\ 0 & \bar{x} \end{bmatrix}$$

where $a, b, x \in GF(p^2)$ are chosen to satisfy

$$a\,\bar{a} + b\,\bar{b} = 1,\ x^5 = 1,\ x \neq \pm 1\,.$$

x must also satisfy $x\,\bar{x} = 1$, that is $x^{p+1} = 1$. Since $p + 1 \equiv 0\,(5)$ this follows automatically.

First we choose x so that $x^5 = 1$, $x \neq \pm 1$ and then we choose a so that $a^2(x-x^{-1})^2 = 2$, assuming also that $p \equiv \pm 1\,(8)$. In other words let $\alpha \in GF(p)$ be such that $\alpha^2 = 2$ and then set $a(x-x^{-1}) = \alpha$. But then we have $a(x-x^{-1}) = \alpha = \alpha^p = a^p(x^p-x^{-p}) = a^p(x^{-1}-x) = -\bar{a}(x-x^{-1})$ and hence $a + \bar{a} = 0$. Therefore, with these choices, (2.18) is a (2, 4, 5) triple.

In a similar fashion the matrices in (2.18) will be a (2, 5, 5) triple if $a, b, x \in GF(p^2)$ are chosen to satisfy $a\,\bar{a} + b\,\bar{b} = 1$, $x^5 = 1$, $x \neq \pm 1$, $a(x-x^{-1}) = \alpha$, where $\alpha \in GF(p)$ is any solution of $u^2 \pm u - 1 = 0$. As a consequence we have the following result.

(2.19). THEOREM.

(a) *If* $p \equiv \pm 1\,(5)$ *then there are* (2, 5, 5) *triples in* $PSl_2(p)$.

(b) *If* $p \equiv \pm 1\,(5)$ *and* $p \equiv \pm 1\,(8)$ *then there are* $(2, 4, 5)$ *triples in* $PSl_2(p)$.

It still remains to prove that we can generate $PSl_2(p)$ by $(2, 5, 5)$ triples or $(2, 4, 5)$ triples.

(2.20). THEOREM. *If* $p \equiv \pm 1\,(5)$ *and* $p \equiv \pm 1\,(8)$ *then any* $(2, 4, 5)$ *triple will generate* $PSl_2(p)$.

Proof. Let (A, B, C) be any $(2, 4, 5)$ triple and let $G = \langle A, B, C \rangle$. Because of the orders of A, B, C it readily follows that $G \not\subseteq A_4, S_4, A_5$.

Suppose $G \subseteq D$, where D is a dihedral group of order $p \pm 1$. Then $BC = CB$, since elements of orders > 2 in a dihedral group commute. Therefore $(BC)^4 = C^4$. But also $(BC)^2 = 1$, and this together with $C^5 = 1$ implies that $C = 1$, a contradiction.

Finally suppose $G \subseteq H$, where H is a maximal solvable subgroup. Recall that we have an extension

$$1 \to \mathbf{Z}_p \to H \xrightarrow{\theta} \mathbf{Z}_{(p-1)/2} \to 1\,.$$

Then $C^4 \in \mathbf{Z}_p$ since $(BC)^2 = 1$ and

$$1 = \theta(BC)^4 = \theta(C^4)\,.$$

From this it follows that the order of C is p, a contradiction. Therefore $G = PSl_2(p)$. Q.e.d.

The generation of $PSl_2(p)$ by $(2, 5, 5)$ triples is more delicate since it is possible to generate A_5 by such triples.

(2.21). THEOREM. *If* $p \equiv \pm 1\,(5)$ *then there are* $(2, 5, 5)$ *triples generating* $PSl_2(p)$.

Proof. First we consider the case $p \equiv 1\,(5)$. The matrices A, B, C in (2.17) will be a $(2, 5, 5)$ triple if

$$-a^2 - bc = 1, \quad x^5 = 1, \quad x \neq \pm 1, \quad a(x - x^{-1}) = \alpha\,,$$

where $\alpha \in GF(p)$ is any solution of $u^2 \pm u - 1 = 0$. In particular $\alpha = x + x^{-1}$ is such a solution. In fact $\alpha^2 + \alpha - 1 = 0$.

As before let $G = \langle A, B, C \rangle$. By arguments similar to those of (2.20) we see that $G \not\subseteq A_4, S_4, D$ or H. To show that G can not be a subgroup of A_5 consider the matrix

$$C^2A = \begin{bmatrix} ax^2 & bx^2 \\ cx^{-2} & -ax^2 \end{bmatrix}.$$

The trace of this matrix is

$$\chi = a(x^2 - x^{-2}) = a(x - x^{-1})(x + x^{-1}) = (x + x^{-1})^2 .$$

Using (2.4) we can show that C^2A does not have order 2, 3, or 5, and this eliminates A_5. Hence $G = PSl_2(p)$ in this case.

For the case $p \equiv -1\ (5)$ we choose matrices A, B, C as in (2.18), where now

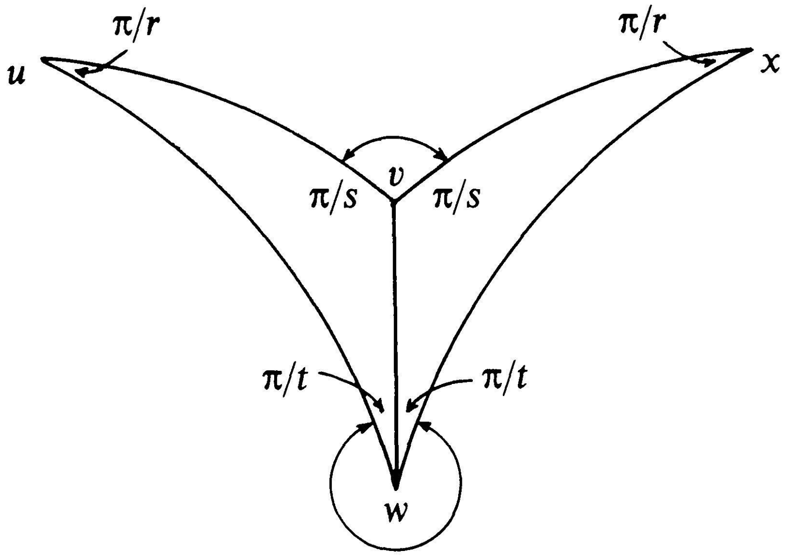
$$a\,\bar{a} + b\,\bar{b} = 1, \quad x^5 = 1, \quad x \neq \pm 1, \quad a(x - x^{-1}) = x + x^{-1} .$$

As in the first case we can show that $\langle A, B, C \rangle = PSl_2(p)$. Q.e.d.

Theorems (2.16), (2.20) and (2.21) now establish half of theorem II in the introduction. The other half follows from the result below.

(2.22). THEOREM. *Suppose G is a finite group and (A, B, C) is an (r, s, t) triple generating G. If $1 \to \Delta \to T(r, s, t) \to G \to 1$ is the associated extension then the genus of H/Δ is* $1 + \dfrac{|G|}{2}\left(1 - \dfrac{1}{r} - \dfrac{1}{s} - \dfrac{1}{t}\right)$.

Proof. A fundamental domain for the action of $T(r, s, t)$ on P, where P is the appropriate plane, consists of two copies of a triangle whose angles are π/r, π/s, π/t (see the diagram)



A, B, C are rotations about u, v, w through angles $2\pi/r$, $2\pi/s$, $2\pi/t$.

The only identifications under the action are: vu gets identified to vx and wu gets identified to wx. It follows that $P/T(r, s, t)$ is the 2 sphere and the branched covering $P/\Delta \to P/T(r, s, t)$ has 3 branch points coming from the vertices u, v, w.

Now notice that Δ is torsion free. This follows from the facts:

(1) the elements of finite order in $T(r, s, t)$ are the conjugates of A, B, C.

(2) elements of finite order in $T(r, s, t)$ map to elements of the same order in G. From this it follows that the orders of the branch points are r, s, t respectively.

Finally we consider the Riemann-Hurwitz formula:

$$\chi(P/\Delta) = |G| \left(\chi(P/T(r, s, t)) - \left(1 - \frac{1}{r}\right) - \left(1 - \frac{1}{s}\right) - \left(1 - \frac{1}{t}\right) \right)$$

i.e.,
$$2 - 2g = |G| \left(\frac{1}{r} + \frac{1}{s} + \frac{1}{t} - 1 \right).$$

Therefore
$$g = 1 + \frac{|G|}{2} \left(1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t} \right)$$
Q.e.d.

§ 3. Conformal Actions on Surfaces of least Genus

If (A, B, C) is an (r, s, t) triple generating $PSl_2(p)$ then we have a short exact sequence

$$1 \to \Delta \to T(r, s, t) \to PSl_2(p) \to 1$$

where Δ is torsion free. Then it follows that $H/T(r, s, t)$ is S^2 and the branched covering $H/\Delta \to H/T(r, s, t)$ has 3 branch points with orders r, s, t.

Conversely we have:

(3.1). Theorem. *If S is a Riemann surface of least genus for $PSl_2(p)$ then $S/PSl_2(p)$ is S^2 and $\pi: S \to S/PSl_2(p)$ has 3 branch points.*

Proof. There exists a short exact sequence $1 \to \Delta \to T(2, 3, p) \to PSl_2(p) \to 1$ arising from a $(2, 3, p)$ triple and consequently

$$\text{genus}\,(H/\Delta) = 1 + \frac{|G|}{2} \left(\frac{1}{6} - \frac{1}{p} \right).$$

Let g = genus (S), h = genus $(S/PSl_2(p))$ and suppose $\pi: S \to S/PSl_2(p)$ has b branch points $x_1, \ldots, x_b$ of respective orders $n_1, \ldots, n_b$. Then the Riemann-Hurwitz formula tells us

$$(3.2). \qquad 2 - 2g = |G| \left(2 - 2h - \sum \left(1 - \frac{1}{n_i}\right)\right).$$

That is $g = 1 + \dfrac{|G|}{2}\left(2h - 2 + \sum\left(1 - \dfrac{1}{n_i}\right)\right)$. Since g is the least genus this leads to the inequality

$$(3.3). \qquad 2h - 2 + \sum\left(1 - \frac{1}{n_i}\right) \leqslant \frac{1}{6} - \frac{1}{p}.$$

From this we immediately see that $h = 0, 1$.

Therefore we suppose that $h = 1$. Since all $n_i \geqslant 2$ this implies that $b = 0$, and hence $PSl_2(p)$ is acting fixed point freely on S with orbit space the torus. But this immediately gives an epimorphism $\mathbf{Z} \oplus \mathbf{Z} \twoheadrightarrow PSl_2(p)$. However, this is a contradiction since G is not abelian. Therefore $h = 0$ and $S/PSl_2(p)$ is a 2-sphere.

To prove that there are 3 branch points put $h = 0$ into (3.3):

$$(3.4) \qquad -2 + \sum_{i=1}^{b}\left(1 - \frac{1}{n_i}\right) \leqslant \frac{1}{6} - \frac{1}{p}.$$

Since $1 - \dfrac{1}{n_i} \geqslant \dfrac{1}{2}$ for all i this gives $b \leqslant 4$. If $b = 0$ we have an unbranched covering $S \to S^2$ with deck transformation group $PSl_2(p)$. But this is clearly a contradiction.

Thus assume $b = 1$. Then we have the regular unbranched covering

$$S - \pi^{-1}(x_1) \to S^2 - \{x_1\}$$

with deck transformation group $PSl_2(p)$. But again this is impossible since $S^2 - \{x_1\} \cong \mathbf{R}^2$.

Next we put $b = 2$ and consider the regular covering

$$S - \pi^{-1}\{x_1, x_2\} \to S^2 - \{x_1, x_2\}.$$

Then we have the exact sequence coming from fundamental groups $1 \to \mathbf{Z} \to \mathbf{Z} \to PSl_2(p) \to 1$, which is again a contradiction.

Finally we suppose $b = 4$. The inequality (3.4) is

(3.5). $$2-\frac{1}{n_1}-\frac{1}{n_2}-\frac{1}{n_3}-\frac{1}{n_4}\leqslant\frac{1}{6}-\frac{1}{p}$$

and is clearly satisfied by $n_1 = n_2 = n_3 = n_4 = 2$. However, this choice of n_i's gives $g = 1$ by (3.2); in other words $PSl_2(p)$ is toroidal. However, no nonabelian finite simple group G can act on $S^1 \times S^1$ because covering space theory implies there are branch points and hence the orbit space is S^2. Hence the induced homomorphism $PSl_2(p) \to \mathrm{Aut}(\mathbf{Z}^2)$ is nontrivial and also has a kernel since $\mathrm{Aut}(\mathbf{Z}^2)$ has no p torsion for $p \geqslant 7$. This contradicts G simple. Therefore this case is excluded and we have

$$2-\frac{1}{n_1}-\frac{1}{n_2}-\frac{1}{n_3}-\frac{1}{n_4}\geqslant 2-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

contradicting (3.5). Q.e.d.

If we let the orders of the 3 branch points be r, s, t then the Riemann-Hurwitz formula is

$$\chi(S) = |PSl_2(p)|\left(2-\left(1-\frac{1}{r}\right)-\left(1-\frac{1}{s}\right)-\left(1-\frac{1}{t}\right)\right).$$

But $|PSl_2(p)| = \dfrac{p(p^2-1)}{2}$ and therefore

$$\text{genus }(S) = 1+\frac{p(p^2-1)}{4}\left(1-\frac{1}{r}-\frac{1}{s}-\frac{1}{t}\right).$$

To take advantage of this formula we must know what sort of branching data (r, s, t) can occur. To this end we quote a very general theorem of Tucker [T].

(3.6). THEOREM. *Suppose G is a finite group acting effectively on a closed orientable surface S by orientation preserving homeomorphisms. If g = genus (S/G) and there are b branch points of orders $n_1, \ldots, n_b$ then G has a presentation of the form*

$$\left\{x_1, y_1, \ldots, x_g, y_g, e_1, \ldots, e_b \mid \prod_{i=1}^{g}[x_i, y_i]e_1 \ldots e_b = e_1^{n_1} = \ldots = e_b^{n_b} = 1, ETC\right\}$$

(3.7). COROLLARY. *If S is a Riemann surface of least genus for $PSl_2(p)$ then there exist integers $r, s, t, \geqslant 2$ so that*

(a) *there is an extension* $1 \to \Delta \to T(r, s, t) \to PSl_2(p) \to 1$;

(b) $\text{genus}\,(PSl_2(p)) = 1 + \dfrac{p(p^2-1)}{4}\left(1 - \dfrac{1}{r} - \dfrac{1}{s} - \dfrac{1}{t}\right).$

If A, B, C are the usual generators of $T(r, s, t)$ then it is in fact true that the orders of A, B, C in $PSl_2(p)$ are r, s, t. Putting (2.22) and (3.7) together gives

(3.8). COROLLARY. *The genus of* $PSl_2(p)$ *is given by*

$$g = \min\left\{1 + \frac{p(p^2-1)}{4}\left(1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t}\right)\right\}$$

where the minimum is taken over all (r, s, t) *for which there exist* (r, s, t) *triples generating* $PSl_2(p)$.

The last step in the determination of the genus is to identify those (r, s, t) which are relevant. This is accomplished in the following manner:

(1) first find all (r, s, t) so that

$$1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t} \leqslant \frac{1}{6} - \frac{1}{d}, \quad \text{assuming } p \geqslant 13\,.$$

(2) then eliminate those triples (r, s, t) corresponding to either spherical or Euclidean triangle groups.

(3) make a comparison of the triples remaining so as to eliminate those with larger genus.

In the following table we give some pertinent data:

TABLE I

(r, s, t)	$1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t}$	type	condition for $1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t} \leqslant \frac{1}{6} - \frac{1}{d}$
$(2, 2, t)$	$-\frac{1}{t}$	spherical	$d \geqslant 6$
$(2, 3, t)$ where $3 < t < 5$	$\frac{1}{6} - \frac{1}{t}$	spherical	$t \leqslant d$
$(2, 3, 6)$	0	euclidean	$t < d$
$(2, 3, t)$ where $t > 7$	$\frac{1}{6} - \frac{1}{t}$	hyperbolic	$t \leqslant d$
$(2, 4, 4)$	0	euclidean	$d > 6$
$(2, 4, 5)$	$\frac{1}{20}$	hyperbolic	$d \geqslant 9$
$(2, 4, 6)$	$\frac{1}{12}$	hyperbolic	$d \geqslant 12$
$(2, 4, 7)$	$\frac{3}{28}$	hyperbolic	$d \geqslant 17$
$(2, 4, 8)$	$\frac{1}{8}$	hyperbolic	$d \geqslant 24$
$(2, 4, 9)$	$\frac{5}{36}$	hyperbolic	$d \geqslant 36$
$(2, 4, 10)$	$\frac{3}{20}$	hyperbolic	$d \geqslant 60$
$(2, 4, 11)$	$\frac{7}{44}$	hyperbolic	$d \geqslant 132$

TABLE I *(suite)*

	(r, s, t)	$1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t}$	*type*	condition for $1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t} \leqslant \frac{1}{6}$
13.	$(2, 4, t)$	$\frac{1}{4} - \frac{1}{t} \geqslant \frac{1}{6}$	hyperbolic	never
14.	$(2, 5, 5)$	$\frac{1}{10}$	hyperbolic	$d \geqslant 15$
15.	$(2, 5, 6)$	$\frac{2}{15}$	hyperbolic	$d \geqslant 30$
16.	$(2, 5, 7)$	$\frac{11}{70}$	hyperbolic	$d \geqslant 105$
17.	$(2, 5, t)$ where $t > 8$	$\frac{3}{10} - \frac{1}{t} > \frac{1}{6}$	hyperbolic	never
18.	$(2, s, t)$	$\frac{1}{2} - \frac{1}{s} - \frac{1}{t} \geqslant \frac{1}{6}$	hyperbolic	never
19.	$(3, 3, 3)$	0	euclidean	$d > 7$
20.	$(3, 3, 4)$	$\frac{1}{12}$	hyperbolic	$d \geqslant 12$
21.	$(3, 3, 5)$	$\frac{2}{15}$	hyperbolic	$d \geqslant 30$
22.	$(3, 3, t)$	$\frac{1}{3} - \frac{1}{t} \geqslant \frac{1}{6}$	hyperbolic	never
23.	$(3, s, t)$ where $t > s > 4$	$\frac{2}{3} - \frac{1}{s} - \frac{1}{t} \geqslant \frac{1}{6}$	hyperbolic	never
24.	(r, s, t) where $t > s > r > 4$	$1 - \frac{1}{r} - \frac{1}{s} - \frac{1}{t} \geqslant \frac{1}{4}$	hyperbolic	never

Examining this table we see that we can eliminate cases 13, 17, 18, 22, 23 and 24 since $\frac{1}{6} - \frac{1}{d}$ will always be less than $1 - 1/r - 1/s - 1/t$. We can also eliminate cases 1, 2, 3, 5 and 19 since these triples are not hyperbolic. Now notice that cases 7, ..., 12 need never be considered since if there are such triples generating $PSl_2(p)$ then there will also be a $(3, 3, 4)$ triple generating $PSl_2(p)$, in which case the genus calculation from the $(3, 3, 4)$ case is at least as small. In a similar fashion we can ignore cases 15, 16 and 21 by comparing them with case 14. Finally, we can use Lemma (2.3) to eliminate case 4. The triples remaining after this will be $(2, 3, p)$, $(2, 3, d)$, $(2, 5, 5)$, $(2, 4, 5)$ and $(3, 3, 4)$. Minimization of the genera for these triples leads directly to the corollary in the introduction.

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