

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 31 (1985)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** NOTE ON LEVI'S PROBLEM WITH DISCONTINUOUS FUNCTIONS  
**Autor:** Coltoiu, Mihnea  
**Kapitel:** §3. The proof of Theorem  
**DOI:** <https://doi.org/10.5169/seals-54571>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 18.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

to be strongly plurisubharmonic if for every  $C^\infty$  real-valued function  $\theta$  with compact support there exists an  $\varepsilon_0 > 0$  such that  $\varphi + \varepsilon\theta$  is plurisubharmonic for  $|\varepsilon| \leq \varepsilon_0$ .

A main result in [3] tells us that the above definition agrees with the usual one as given in [6].

Let us also recall that a complex space  $X$  is said to be 1-convex if there exist:

- i) a compact analytic set  $S \subset X$  with  $\dim_x S > 0$  for any  $x \in S$ ,
- ii) a Stein space  $Y$ , a finite set  $A \subset Y$  and a proper holomorphic map  $p: X \rightarrow Y$  inducing a biholomorphism  $X \setminus S \cong Y \setminus A$  and which satisfies  $p_* \mathcal{O}_X \cong \mathcal{O}_Y$ .

$S$  is called the exceptional set of  $X$  and  $Y$  the Remmert reduction of  $X$ .

*Remark.* Using the analytic version of Chow's lemma (Hironaka [5]) it was proved in [2] that any 1-convex space  $X$  carries a strongly plurisubharmonic exhaustion function  $\varphi: X \rightarrow [-\infty, \infty)$ , i.e. the converse of Theorem 1 holds too.

### § 3. THE PROOF OF THEOREM

We shall apply Andreotti-Grauert's technique [1] with suitable modifications required by the upper semicontinuity. Throughout this section  $\mathcal{F}$  will denote a coherent sheaf on  $X$  and  $X_c = \{x \in X \mid \varphi(x) < c\}$ .

To prove Theorem 1 we need some lemmas.

LEMMA 1. *For any  $c \in \mathbf{R}$  there exists  $\varepsilon > 0$  such that restriction map  $H^1(X_{c+\varepsilon}, \mathcal{F}) \rightarrow H^1(X_{c+\varepsilon'}, \mathcal{F})$  is surjective for any  $0 \leq \varepsilon' \leq \varepsilon$ .*

*Proof.* We may assume  $c = 0$ . Set  $K = \overline{\{\varphi < 1\}}$  and let  $\{U_1, \dots, U_m\}$  be a covering of  $K$  with Stein open sets,  $U_i \subset \subset X$  and  $h_i \in C_0^\infty(U_i)$ ,  $h_i \geq 0$  such that  $\varphi - \sum_{i=1}^r h_i$  is strongly plurisubharmonic for  $r = 1, \dots, m$  and  $\sum_{i=1}^m h_i > 0$  on  $K$ . Choose  $\alpha > 0$  such that  $\sum_{i=1}^m h_i(x) \geq \alpha$  for any  $x \in K$  and take  $0 < \varepsilon < \min(\alpha, 1)$ . We shall prove that this  $\varepsilon$  satisfies the conditions required in Lemma 1.

For any  $0 \leq \varepsilon' \leq \varepsilon$  we set  $X_{\varepsilon'}^r = \{x \in X \mid \varphi(x) < \varepsilon' + h_1(x) + \dots + h_r(x)\}$  for  $r = 0, \dots, m$  (by definition  $X_{\varepsilon'}^0 = X_{\varepsilon'}$ ).

We make the following remark: for any  $0 \leq \varepsilon' \leq \varepsilon$  we have  $X_\varepsilon \subset X_{\varepsilon'}^m$ . Indeed, let  $x \in X$  such that  $\varphi(x) < \varepsilon$ . In particular  $\varphi(x) < 1$ , hence  $x \in K$ . From the definition of  $\alpha$  it follows that  $\sum_{i=1}^m h_i(x) \geq \alpha$  and from the inequalities

$$\varphi(x) < \varepsilon < \alpha \leq \sum_{i=1}^m h_i(x) \leq \varepsilon' + \sum_{i=1}^m h_i(x) \text{ we get } x \in X_{\varepsilon'}^m.$$

Due to this remark Lemma 1 will be proved if we prove that the restriction map  $H^1(X_{\varepsilon'}^m, \mathcal{F}) \rightarrow H^1(X_{\varepsilon'}, \mathcal{F})$  is surjective for any  $0 \leq \varepsilon' \leq \varepsilon$ . The inclusions  $X_{\varepsilon'} = X_{\varepsilon'}^0 \subset X_{\varepsilon'}^1 \subset \dots \subset X_{\varepsilon'}^m$  show that it suffices to prove that the restrictions  $H^1(X_{\varepsilon'}^{r+1}, \mathcal{F}) \rightarrow H^1(X_{\varepsilon'}^r, \mathcal{F})$  are surjective for  $r = 0, \dots, m-1$ . If we set

$$V_{\varepsilon'}^{r+1} = \{x \in U_{r+1} \mid \varphi(x) < \varepsilon' + h_1(x) + \dots + h_{r+1}(x)\}$$

then  $V_{\varepsilon'}^{r+1}$  and  $X_{\varepsilon'}^r \cap V_{\varepsilon'}^{r+1}$  are Stein open sets. On the other hand  $X_{\varepsilon'}^{r+1} \setminus X_{\varepsilon'}^r \subset \text{supp}(h_{r+1}) \subset U_{r+1}$  and so  $X_{\varepsilon'}^{r+1} = X_{\varepsilon'}^r \cup V_{\varepsilon'}^{r+1}$ . From the Mayer-Vietoris exact sequence:

$$H^1(X_{\varepsilon'}^{r+1}, \mathcal{F}) \rightarrow H^1(X_{\varepsilon'}^r, \mathcal{F}) \oplus H^1(V_{\varepsilon'}^{r+1}, \mathcal{F}) \rightarrow H^1(X_{\varepsilon'}^r \cap V_{\varepsilon'}^{r+1}, \mathcal{F})$$

it follows that the restriction map  $H^1(X_{\varepsilon'}^{r+1}, \mathcal{F}) \rightarrow H^1(X_{\varepsilon'}^r, \mathcal{F})$  is surjective and so Lemma 1 is proved.

LEMMA 2. For any  $\alpha \leq \beta$  the restriction map  $H^1(X_\beta, \mathcal{F}) \rightarrow H^1(X_\alpha, \mathcal{F})$  is surjective.

*Proof.* Set  $M(\alpha) = \{\delta \geq \alpha \mid \text{for any } \alpha \leq \gamma \leq \delta \text{ the restriction map}$

$$H^1(X_\delta, \mathcal{F}) \rightarrow H^1(X_\gamma, \mathcal{F}) \text{ is surjective}\}.$$

From Lemma 1 and Lemma [1, p. 241] we deduce that  $M(\alpha) = [\alpha, \infty)$  which proves Lemma 2.

LEMMA 3. For any  $\alpha \in \mathbf{R}$   $H^1(X_\alpha, \mathcal{F})$  has finite dimension.

*Proof.* Choose  $\beta > \alpha$  such that  $\bar{X}_\alpha \subset X_\beta$ . From Lemma 2 the restriction map  $H^1(X_\beta, \mathcal{F}) \rightarrow H^1(X_\alpha, \mathcal{F})$  is surjective and from [1, p. 240]

$$\dim_{\mathbf{C}} H^1(X_\alpha, \mathcal{F}) < \infty.$$

LEMMA 4. For any  $c \in \mathbf{R}$  there exists  $\varepsilon > 0$  such that the restriction map  $\Gamma(X_{c+\varepsilon}, \mathcal{F}) \rightarrow \Gamma(X_{c+\varepsilon'}, \mathcal{F})$  has dense image for any  $0 \leq \varepsilon' \leq \varepsilon$ .

*Proof.* We may assume  $c = 0$  and choose  $\varepsilon > 0$  as in Lemma 1. Exactly as in the proof of Lemma 1 it suffices to prove that the restriction map  $\Gamma(X_{\varepsilon'}^{r+1}, \mathcal{F}) \rightarrow \Gamma(X_{\varepsilon'}^r, \mathcal{F})$  has dense image for  $r = 0, \dots, m - 1$ .

Consider the Mayer-Vietoris exact sequence:

$$\begin{aligned} \Gamma(X_{\varepsilon'}^{r+1}, \mathcal{F}) &\rightarrow \Gamma(X_{\varepsilon'}^r, \mathcal{F}) \oplus \Gamma(V_{\varepsilon'}^{r+1}, \mathcal{F}) \xrightarrow{\alpha} \Gamma(X_{\varepsilon'}^r \cap V_{\varepsilon'}^{r+1}, \mathcal{F}) \\ &\rightarrow H^1(X_{\varepsilon'}^{r+1}, \mathcal{F}) \end{aligned}$$

Since  $(X_{\varepsilon'}^r \cap V_{\varepsilon'}^{r+1}, V_{\varepsilon'}^{r+1})$  is a Runge pair it follows that  $\alpha$  has dense image. On the other hand, applying Lemma 3 to the function

$$\varphi - \varepsilon' - h_1 - \dots - h_{r+1}$$

we deduce that  $H^1(X_{\varepsilon'}^{r+1}, \mathcal{F})$  has finite dimension, in particular it is separated, hence  $\alpha$  has closed image. Consequently  $\alpha$  is surjective. From the open mapping theorem it follows easily that the restriction map

$$\Gamma(X_{\varepsilon'}^{r+1}, \mathcal{F}) \rightarrow \Gamma(X_{\varepsilon'}^r, \mathcal{F})$$

has dense image and so Lemma 4 is proved.

**LEMMA 5.** *For any  $\alpha \leq \beta$  the restriction map  $\Gamma(X_{\beta}, \mathcal{F}) \rightarrow \Gamma(X_{\alpha}, \mathcal{F})$  has dense image.*

*Proof.* Lemma 5 is an immediate consequence of Lemma 4 and of Lemma [1, p. 246].

**LEMMA 6.** *For any  $c \in \mathbf{R}$  there exists  $\varepsilon > 0$  such that the restriction map  $H^1(X_{c+\varepsilon}, \mathcal{F}) \rightarrow H^1(X_{c+\varepsilon'}, \mathcal{F})$  is bijective for any  $0 \leq \varepsilon' \leq \varepsilon$ .*

*Proof.* We may assume  $c = 0$  and choose  $\varepsilon > 0$  as in Lemma 1. Due to the inclusions  $X_{\varepsilon'} \subset X_{\varepsilon} \subset X_{\varepsilon}^m$  and using Lemma 2 it follows that it suffices to show that the restriction map  $H^1(X_{\varepsilon}^m, \mathcal{F}) \rightarrow H^1(X_{\varepsilon'}, \mathcal{F})$  is bijective. The inclusions  $X_{\varepsilon'} = X_{\varepsilon'}^0 \subset X_{\varepsilon'}^1 \subset \dots \subset X_{\varepsilon'}^m$  show that it is enough to prove that the restrictions  $H^1(X_{\varepsilon'}^{r+1}, \mathcal{F}) \rightarrow H^1(X_{\varepsilon'}^r, \mathcal{F})$  are bijective for  $r = 0, \dots, m - 1$ .

Consider the Mayer-Vietoris exact sequence:

$$\begin{aligned} \Gamma(X_{\varepsilon'}^r, \mathcal{F}) \oplus \Gamma(V_{\varepsilon'}^{r+1}, \mathcal{F}) &\rightarrow \Gamma(X_{\varepsilon'}^r \cap V_{\varepsilon'}^{r+1}, \mathcal{F}) \rightarrow H^1(X_{\varepsilon'}^{r+1}, \mathcal{F}) \\ &\rightarrow H^1(X_{\varepsilon'}^r, \mathcal{F}) \oplus H^1(V_{\varepsilon'}^{r+1}, \mathcal{F}) \rightarrow H^1(X_{\varepsilon'}^r \cap V_{\varepsilon'}^{r+1}, \mathcal{F}) \end{aligned}$$

As remarked in the proof of Lemma 4 the map

$$\Gamma(X_{\varepsilon'}^r, \mathcal{F}) \oplus \Gamma(V_{\varepsilon'}^{r+1}, \mathcal{F}) \rightarrow \Gamma(X_{\varepsilon'}^r \cap V_{\varepsilon'}^{r+1}, \mathcal{F})$$

is surjective. Since

$$H^1(V_{\varepsilon'}^{r+1}, \mathcal{F}) = H^1(X_{\varepsilon'}^r \cap V_{\varepsilon'}^{r+1}, \mathcal{F}) = 0$$

it follows that the restriction map

$$H^1(X_{\varepsilon'}^{r+1}, \mathcal{F}) \rightarrow H^1(X_{\varepsilon'}^r, \mathcal{F})$$

is bijective and so Lemma 6 is proved.

LEMMA 7. For any  $\alpha \leq \beta$  the restriction map  $H^1(X_\beta, \mathcal{F}) \rightarrow H^1(X_\alpha, \mathcal{F})$  is bijective.

*Proof.* Set  $M(\alpha) = \{\delta \geq \alpha \mid \text{for any } \alpha \leq \gamma \leq \delta \text{ the restriction map}$

$$H^1(X_\delta, \mathcal{F}) \rightarrow H^1(X_\gamma, \mathcal{F}) \text{ is bijective}\}$$

and let  $\alpha_0 = \sup M(\alpha)$ .

From Lemma 2 it follows that if  $\delta \in M(\alpha)$  then  $[\alpha, \delta] \subset M(\alpha)$ , consequently  $[\alpha, \alpha_0] \subset M(\alpha)$ . To prove Lemma 7 we have to show that  $\alpha_0 = \infty$ . Suppose that  $\alpha_0 < \infty$ . From Lemma 5 and Lemma [1, p. 250] we deduce that  $\alpha_0 \in M(\alpha)$ . From Lemma 6 there exists  $\varepsilon > 0$  such that  $\alpha_0 + \varepsilon \in M(\alpha)$ . This contradicts the definition of  $\alpha_0$ , and so Lemma 7 is proved.

We are now in a position to prove Theorem 1. Choose  $\alpha \in \mathbf{R}$  and take  $\alpha = \alpha_0 < \alpha_1 < \dots < \alpha_n < \dots$  an increasing sequence of real numbers tending to  $\infty$ . By Lemma 7 the restriction map  $H^1(X_{\alpha_{n+1}}, \mathcal{F}) \rightarrow H^1(X_{\alpha_n}, \mathcal{F})$  is bijective and by Lemma 5 the restriction map  $\Gamma(X_{\alpha_{n+1}}, \mathcal{F}) \rightarrow \Gamma(X_{\alpha_n}, \mathcal{F})$  has dense image. It follows then from Lemma [1, p. 250] that the restriction map  $H^1(X, \mathcal{F}) \rightarrow H^1(X_\alpha, \mathcal{F})$  is also bijective and from Lemma 3  $H^1(X, \mathcal{F})$  has finite dimension. Theorem V. in [6] tells us that  $X$  is 1-convex, as required.