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A NOTE ON LEVI'S PROBLEM WITH DISCONTINUOUS FUNCTIONS

by Mihnea Coltoiu

§ 1. Introduction

In [3] Fornaess and Narasimhan proved that a complex space X which carries a strongly plurisubharmonic exhaustion function $\varphi : X \to \mathbf{R}$ is a Stein space. It is a remarkable fact that φ is supposed only upper semicontinuous.

A natural question which arises when we consider the Levi problem with upper semicontinuous functions is the following: what would happen if we allowed φ to take on the value $-\infty$. Simple examples (compact complex spaces, the blowing up of $\mathbf{C}^n$ at the origine...) show us that X is not necessarily Stein. The best result one might hope to obtain is X being 1-convex.

The aim of this short note is to give an affirmative answer to this question, hence to prove the following theorem conjectured by Fornaess and Narasimhan:

Theorem 1. *Let X be a complex space which admits a strongly plurisubharmonic exhaustion function $\varphi : X \to [-\infty, \infty)$. Then X is 1-convex.*

If φ is supposed real-valued it follows easily, from the maximum principle, that the exceptional set of X is empty, hence X is Stein. This is exactly Fornaess-Narasimhan's theorem.

§ 2. Preliminaries

All complex spaces are assumed to be reduced and countable at infinity.

An upper semicontinuous function $\varphi : X \to [-\infty, \infty)$ is called plurisubharmonic if for every holomorphic map $\tau : W \to X$ (W = the unit disc in $\mathbf{C}$) it follows that $\varphi \circ \tau$ is subharmonic on W (possibly $\equiv -\infty$). φ is said to be strongly plurisubharmonic if for every C^∞ real-valued function θ with compact support there exists an $\varepsilon_0 > 0$ such that $\varphi + \varepsilon\theta$ is plurisubharmonic for $|\varepsilon| \leqslant \varepsilon_0$.

A main result in [3] tells us that the above definition agrees with the usual one as given in [6].

Let us also recall that a complex space X is said to be 1-convex if there exist:

i) a compact analytic set $S \subset X$ with $\dim_x S > 0$ for any $x \in S$,

ii) a Stein space Y, a finite set $A \subset Y$ and a proper holomorphic map $p: X \to Y$ inducing a bilholomorphism $X \backslash S \cong Y \backslash A$ and which satisfies $p_* \mathcal{O}_X \cong \mathcal{O}_Y$.

S is called the exceptional set of X and Y the Remmert reduction of X.

Remark. Using the analytic version of Chow's lemma (Hironaka [5]) it was proved in [2] that any 1-convex space X carries a strongly plurisubharmonic exhaustion function $\varphi: X \to [-\infty, \infty)$, i.e. the converse of Theorem 1 holds too.

§ 3. The proof of Theorem

We shall apply Andreotti-Grauert's technique [1] with suitable modifications required by the upper semicontinuity. Throughout this section $\mathcal{F}$ will denote a coherent sheaf on X and $X_c = \{x \in X \mid \varphi(x) < c\}$.

To prove Theorem 1 we need some lemmas.

Lemma 1. *For any* $c \in \mathbf{R}$ *there exists* $\varepsilon > 0$ *such that is restriction map* $H^1(X_{c+\varepsilon}, \mathcal{F}) \to H^1(X_{c+\varepsilon'}, \mathcal{F})$ *is surjective for any* $0 \leqslant \varepsilon' \leqslant \varepsilon$.

Proof. We may assume $c = 0$. Set $K = \overline{\{\varphi < 1\}}$ and let $\{U_1, \ldots, U_m\}$ be a covering of K with Stein open sets, $U_i \subset\subset X$ and $h_i \in C_0^\infty(U_i)$, $h_i \geqslant 0$ such that $\varphi - \sum_{i=1}^{r} h_i$ is strongly plurisubharmonic for $r = 1, \ldots, m$ and $\sum_{i=1}^{m} h_i > 0$ on K. Choose $\alpha > 0$ such that $\sum_{i=1}^{m} h_i(x) \geqslant \alpha$ for any $x \in K$ and take $0 < \varepsilon < \min(\alpha, 1)$. We shall prove that this ε satisfies the conditions required in Lemma 1.

For any $0 \leqslant \varepsilon' \leqslant \varepsilon$ we set $X_{\varepsilon'}^r = \{x \in X \mid \varphi(x) < \varepsilon' + h_1(x) + \ldots + h_r(x)\}$ for $r = 0, \ldots, m$ (by definition $X_{\varepsilon'}^0 = X_{\varepsilon'}$).