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$$\begin{array}{ccccccc}
 H^2(\mathbf{Z}/p^2; \Lambda_1) & \rightarrow & H^2(\mathbf{Z}/p^2; M) & \rightarrow & 0 \\
 \downarrow \Delta = i_*(\Lambda_1) & & \downarrow i_*(M) & & \\
 H^2(\mathbf{Z}/p; \varphi_2 \Lambda_2) & \xrightarrow{\lambda_*^k} & H^2(\mathbf{Z}/p; \Lambda_1) & \rightarrow & H^2(\mathbf{Z}/p; M) & \rightarrow & 0
 \end{array}$$

where  $\Delta$  is the diagonal map  $\mathbf{Z}_p \rightarrow p(\mathbf{Z}/p)$ . Hence to eliminate  $M_6(k)$ , it suffices to show  $\text{Im}(\Delta) \subset \text{Im}(\lambda_*^k)$ . Let  $e$  denote a column  $p$ -vector consisting of all 1's, according to the proof of (2.5) we must find an  $\bar{x}_k$ ,  $1 \leq k \leq p - 1$  so that  $C_{p,k} \cdot \bar{x}_k = C_{p,1}^k \cdot \bar{x}_k = e$ . We do this inductively

on  $k$ . For example,  $\bar{x}_1 = \begin{bmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ p \end{bmatrix}$ , as can easily be checked. Inductively we

define

$$\bar{x}_k(i) = \begin{cases} 1 & i = 1 \\ \bar{x}_k(i-1) + \bar{x}_{k-1}(i) & i > 1. \end{cases}$$

Clearly  $C_{p,1} \cdot \bar{x}_k = \bar{x}_{k-1}$ , for all coordinates except possibly the first; we must show  $\bar{x}_k(p) \equiv 0 \pmod{p}$ . But a comparison of the  $\bar{x}_k$ 's with Pascal's triangle convinces one that

$$\bar{x}_k(p) = \binom{p-1+k}{p-1} \equiv \binom{k-1}{p-1} \binom{1}{0} \equiv 0 \pmod{p},$$

since  $k - 1 < p - 1$ .

We leave it for the reader to check that the restriction maps for  $M_7(k)$  and  $M_9(k)$  are non-trivial.

### § 3. $\mathbf{Z}/4$ -MANIFOLDS

In this section, we consider the case  $p = 2$ . For convenience, we change the notation slightly and write  $M_7$  for  $M_6(1)$  and  $M_i$  for  $M_i(0)$ ,  $i = 6, 8, 9$ . According to (2.7), the indecomposable  $\mathbf{Z}/4$ -lattices that carry special classes are  $M_1$ ,  $M_4$  and  $M_9$ . It is easy to see  $M_i$  is faithful if and only if

$i = 3, 5, 6, 7, 8, 9$ . Hence if  $M = \sum m_i M_i$  is an arbitrary  $\mathbf{Z}/4$ -lattice then  $M$  is a faithful representation carrying a special class if and only if the multiplicities  $m_i$  satisfy the inequalities:

$$m_1 + m_4 + m_9 > 0$$

(3.0)

$$m_3 + m_5 + m_6 + m_7 + m_8 + m_9 > 0.$$

Since the multiplicities are a complete set of isomorphism invariants in the case  $p = 2$  (see section 1) one can use the conditions (3.0) to show:

(3.1) THEOREM. *If  $L_n(m)$  denotes the number of isomorphism classes of  $n$ -dimensional  $\mathbf{Z}/m$ -lattices that carry special classes, then:*

$$L_n(4) = \sum_{j=2}^{n-1} \left( a_j - \left[ \frac{j}{2} \right] - 1 \right) + \sum_{j=[n]_2+2}^{n-2} (a_j - a_{j-1} - 1) + \sum_{j=[n]_4}^{n-4} (a_j - a_{j-2} - a_{j-4} + a_{j-6})$$

where  $[k]_p$  denotes the reduction of  $k$  modulo  $p$ ,  $[k]$  denote the largest integer  $\leq k$  and the  $a_j$ 's are given by

$$P(t) = \sum_{j=0}^{\infty} a_j t^j = \frac{1}{(1-t)(1-t^2)^2(1-t^3)^2(1-t^4)^3}$$

In particular, the number of  $n$ -dimensional  $\mathbf{Z}/4$ -manifolds is at least  $L_n(4)$ .

*Proof.* If  $Q(t)$  is a power series, let  $\text{coef}(n, Q(t))$  denote the coefficient of  $t^n$  in  $Q(t)$ . The number  $L_n(4)$  counts the number of ways of writing

$$n = m_1 + m_2 + 2(m_3 + m_4) + 3(m_5 + m_8) + 4(m_6 + m_7 + m_9)$$

where the  $m_i$ 's satisfy (3.0). If  $m_1 > 0$  there is a contribution:

$$\sum_{m_1=1}^{n-2} \text{coef}(n - m_1, P(t)) - \left( \left[ \frac{n - m_1}{2} \right] + 1 \right)$$

where  $\left[ \frac{n - m_1}{2} \right] + 1$  is the number of ways of expressing  $n - m_1$  as a combination of 1's ( $M_2$ ) and 2's ( $M_4$ ) (not permitted by (3.0)). Reindexing gives the first term for  $L_n(4)$ .

Similarly, if  $m_1 = 0, m_4 > 0$  there is a contribution:

$$\sum_{m_4} \text{coef}(t^{n-2m_4}(1-t)P(t)) - 1$$

where 1 is subtracted to omit choosing  $m_2$  alone. Finally, if  $m_1 = m_4 = 0$ , we have:

$$\sum_{m_9} \text{coef}(t^{n-4m_9}, (1-t^2)(1-t^4)P(t)).$$

The coefficients of  $(1-t)P(t)$  and  $(1-t^2-t^4-t^6)P(t)$  are easily expressible in terms of the  $a_j$ 's and the result follows.

*Remark.* In order for a  $\mathbb{Z}/p$ -lattice to carry a special class, the multiplicity of the trivial representation must be non-zero. Topologically this is reflected in the fact that a  $\mathbb{Z}/p$ -manifold fibers over a circle. This is already false for a 4-dimensional  $\mathbb{Z}/4$ -manifold as the following example shows.

*Example.*  $L_4(4) = 6$ . The multiplicities of the indecomposables in these 4-dimensional  $\mathbb{Z}/4$ -lattices are given by:

notation of [2]	$m_1$	$m_2$	$m_3$	$m_4$	$m_5$	$m_6$	$m_7$	$m_8$	$m_9$
07/02/02	1				1				
12/01/04	1							1	
12/01/02	1	1	1						
07/02/01	2		1						
12/01/03			1	1					
12/01/06									1

where the first column gives the label of these “ $\mathbb{Z}$ -classes” from the table of the four-dimensional crystallographic groups in [2]. In fact, as these tables indicate, there is precisely one  $\mathbb{Z}/4$ -manifold corresponding to each  $\mathbb{Z}/4$ -lattice, hence there are exactly 6 4-dimensional  $\mathbb{Z}/4$ -manifolds.

*Remark.* Recall that if  $p < 23$ , the field  $\mathbb{Q}(e^{2\pi i/p})$  has class number one. This fact, along with the work of Charlap [4], shows that the number of  $n$ -dimensional  $\mathbb{Z}/p$ -manifolds is exactly  $L_n(p)$ ,  $p < 23$ . This number is readily computable, as Charlap [4, p. 30] remarks, and the precise formula is:

$$(3.2) \quad L_n(p) = \sum_{j=p-1}^{n-1} \left( \left[ \frac{j}{p-1} \right] - \left\langle \frac{j}{p} \right\rangle + 1 \right)$$

where  $\langle k \rangle$  denotes the smallest integer  $\geq k$ . In particular,  $L_p(p) = 1$ ,  $L_n(p) = 0$ ,  $p > n$ , and when  $p = 2$

$$(3.3) \quad L_n(2) = \binom{n}{2}^2 + \binom{n}{2} - 1 + \frac{[n]_2}{4}$$

using the notation of (3.1).

One can easily construct the following table of values of  $L_n(p)$ :

$n$	2	3	4	5	6
$p$					
2	1	3	5	8	11
3		1	2	3	4
5				1	2

Hence 14 of the 74 4-dimensional flat manifolds have cyclic holonomy  $\leq 5$ . (Furthermore, 26 have holonomy the Klein 4-group.) We describe analogous facts in dimension 5 below.

We let  $SH^2(H, M)$  denote the set of special classes in  $H^2(H, M)$ . If  $H$  is a cyclic  $p$ -group and  $i: \mathbf{Z}_p \hookrightarrow H$  is the inclusion of the subgroup of order  $p$ , then

$$SH^2(H, M) = H^2(H, M) - \ker(i^*).$$

If  $N$  (resp.  $Z$ ) denotes the normalizer (resp. the centralizer) of  $H$  in  $\text{Aut}(M)$ , there is an exact sequence (see [15, p. 50])

$$0 \rightarrow Z \rightarrow N \rightarrow \text{Aut}(H).$$

We conjecture:

*Conjecture.* If  $\mathbf{Z}[e^{2\pi i/p^k}]$  is a unique factorization domain for,  $1 \leq k \leq n$ , then  $N$  acts transitively on  $SH^2(\mathbf{Z}/p^n; M)$  for any  $H$ -lattice  $M$ .

The case  $n = 1$  of the Conjecture follows from Charlap [4]. Class number tables shows that the  $n = 2$  case applies to  $p = 2, 3, 5$ , the  $n = 3$  case to 2, 3 and the  $n = 4$  case to  $p = 2$ . This conjecture implies that the lower bound of (3.1) is exact.

We mention that the multiplicities of the indecomposables in the 5-dimensional  $\mathbf{Z}/4$ -lattices that admit special classes are given by:

	$m_1$	$m_2$	$m_3$	$m_4$	$m_5$	$m_6$	$m_7$	$m_8$	$m_9$
	1								1
	1						1		
*						1			
	1	1			1				
*	1	1						1	
	1		1	1					
*	1		2						
*	1	2	1						
*	3							1	
	2				1				
*	2	1	1						
	3		1						
*		1	1	1					
				1	1				
*				1				1	
*		1							1

Those lattices that are starred clearly satisfy the Conjecture.

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