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The splintering of these genera into isomorphism classes has been analyzed by Reiner [13]. One can, of course, replace R_1, R_2 by ideal classes in these rings and Λ_1 by $E(\mathfrak{a})$ (cf. section 1) where $\mathfrak{a}$ is an ideal class in R_1. There is an additional invariant lying in a quotient of the group of units of a certain finite ring and, if $p \equiv 1 \pmod 4$ a certain quadratic residue character mod p can also appear as an invariant. The precise result is Theorem 7.3 of [13]. We will require only the observation [13, p. 494] that if $p = 2, 3$ there are no further invariants, i.e. each genus of an indecomposable is a single isomorphism class. In the case $p = 5$ already, although the class number of $\mathbf{Q}(e^{2\pi i/m})$ is one for $m = 5, 25$, the 21 genera of indecomposables split up into 40 isomorphism classes. Hence already the further isomorphism invariants mentioned above exert an influence.

§ 2. Cohomology, restrictions and special classes

If H is a finite group, M an H-lattice then: $H^i(H, M) \cong \oplus_p H^i(H, M_p)$, where p ranges over the primes dividing the order of H [3, p. 84]. Hence if M and M' are locally isomorphic, $H^i(H, M) \cong H^i(H, M')$; so the cohomology of an H-lattice depends only on its genus.

We recall the cohomology of a cyclic group $\mathbf{Z}/n = \langle\sigma\rangle$ [3, p. 58]. We write $N = 1 + \sigma \ldots + \sigma^{n-1}$ and $D = 1 - \sigma$. If M is a $\mathbf{Z}/n$-module, then

$$H^0(\mathbf{Z}/n, M) = M^\sigma$$

$$H^{2i-1}(\mathbf{Z}/n, M) = {}_N M / D \cdot M$$

$$H^{2i}(\mathbf{Z}/n, M) = M^\sigma / N \cdot M$$

for all $i \geqslant 1$, where M^σ denotes σ-invariants and ${}_N M = \{x \in M : Nx = 0\}$. From these remarks it is easy to compute the cohomology of the indecomposable $\mathbf{Z}/p$-lattices described in section 1.

(2.1) Proposition. *The following table describes the cohomology of the indecomposable $\mathbf{Z}/p$-lattices:*

M	rank	H^0	H^1	H^2
$\mathbf{1}$	1	$\mathbf{Z}$	0	$\mathbf{Z}/p$
α	$p - 1$	0	$\mathbf{Z}/p$	0
β	p	$\mathbf{Z}$	0	0

Similarly one can easily compute the cohomology and restriction of the first four $\mathbf{Z}/p^2$-lattices of (1.3).

(2.2) PROPOSITION. *If* $[M]_p$ *denotes the restriction of the* $\mathbf{Z}/p^2$*-lattice* M *to the subgroup of order* p*, we have the following table.*

M	rank	H^0	H^1	H^2	$[M]_p$
$M_1 = \mathbf{1}$	1	$\mathbf{Z}$	0	$\mathbf{Z}/p^2$	$\mathbf{1}$
$M_2 = R_1$	$p-1$	0	$\mathbf{Z}/p$	0	$(p-1)\mathbf{1}$
$M_3 = R_2$	p^2-p	0	$\mathbf{Z}/p$	0	$p\alpha$
$M_4 = \Lambda_1$	p	$\mathbf{Z}$	0	$\mathbf{Z}/p$	$p\mathbf{1}$

Furthermore, $\Lambda_2 = \mathbf{Z}[\mathbf{Z}/p^2]$, the regular representation of $\mathbf{Z}/p^2$, satisfies $H^0 = \mathbf{Z}$, $H^1 = 0 = H^2$ and $[\Lambda_2]_\beta = p\beta$.

Proof. It suffices to observe that $M_2 = p^*(\alpha)$ and $M_4 = p^*(\beta)$, where $p: \mathbf{Z}/p^2 \to \mathbf{Z}/p$ is the natural projection, and M_3 fits into a short exact sequence:

$$0 \to M_4 \to \Lambda_2 \to M_3 \to 0\,.$$

The last remark follows from the freeness of Λ_2.

To complete the table for the modules M_i, $i \geqslant 5$, we have the following lemma:

(2.3) LEMMA. *If* L *is a* $\mathbf{Z}/p$*-lattice,* $\alpha \in L$*, then the extension* $M = (L, \alpha)$ *defined by (1.2) satisfies:*

$$H^2(C; M) \cong \operatorname{coker}\big(x_*: H^2(C; \varphi_2\Lambda_2) \to H^2(C; L)\big)$$

where C *is either* $\mathbf{Z}/p^2$ *or* $\mathbf{Z}/p$.

Proof. The diagram (1.2) induces

$$\begin{array}{ccccccccc}
\to & H^1(C, R_2) & \xrightarrow{\delta} & H^2(C, \varphi_2\Lambda_2) & \to & 0 & & \\
 & \downarrow & & \downarrow x_* & & & & \\
\to & H^1(C, R_2) & \xrightarrow[\delta]{} & H^2(C, L) & \to & H^2(C, M) & \to & 0
\end{array}$$

where the zeros follow from (2.2). An easy diagram-chase completes the proof.

(2.4) PROPOSITION. *The following table describes the cohomology of the indecomposable* $\mathbf{Z}/p^2$*-lattices* $M_i, i \geqslant 5$:

M	rank	H^0	H^1	H^2
M_5	$p^2 - p + 1$	$\mathbf{Z}$	0	$\mathbf{Z}/p$
$M_6(0)$	p^2	$\mathbf{Z}$	0	0
$M_6(k)$	p^2	$\mathbf{Z}$	$\mathbf{Z}/p$	$\mathbf{Z}/p$
$M_7(k)$	$p^2 + 1$	$\mathbf{Z} \oplus \mathbf{Z}$	0	$\mathbf{Z}/p \oplus \mathbf{Z}/p$
$M_8(0)$	$p^2 - 1$	0	$\mathbf{Z}/p^2$	0
$M_8(k)$	$p^2 - 1$	0	$\mathbf{Z}/p \oplus \mathbf{Z}/p$	0
$M_9(k)$	p^2	$\mathbf{Z}$	$\mathbf{Z}/p$	$\mathbf{Z}/p$

Proof. Since

$$H^0(\mathbf{Z}/p^2; R_2) = 0, H^0(\mathbf{Z}/p^2; (L, x)) \cong H^0(\mathbf{Z}/p^2; L)$$

and these can be read off from (2.2). The groups H^2 are computed by (2.3). We work out one example in detail. Consider $M_6(k), 0 \leqslant k \leqslant p - 1$, so that $L = \Lambda_1$. If we identify $\varphi_2\Lambda_2$ with Λ_1 then the generator:

$$1 + x + ... + x^{p-1} \in H^2(\mathbf{Z}/p^2, \varphi_2\Lambda_2)$$

is sent by λ_*^k, $1 \leqslant k \leqslant p - 1$, to

$$(1-x)^k(1+x+...+x^{p-1}) = (1-x)^{k-1} \cdot 0 = 0$$

in $H^2(\mathbf{Z}/p^2; \Lambda_1)$. If $k = 0$, then the map is an isomorphism. Hence $H^2(\mathbf{Z}/p^2; M_6(0)) = 0$ and $H^2(\mathbf{Z}/p^2, M_6(k)) = \mathbf{Z}/p, k \geqslant 1$.

The groups $H^1(M_i)$ can be read off the long exact cohomology sequence of the bottom row of (1.2).

Remark. It follows from (2.2) that $M_6(0)$ is the genus of the regular representation.

We now record the restrictions of the modules M_i to the subgroup of order p.

(2.5) PROPOSITION. *The* $\mathbf{Z}/p$*-cohomology and the restrictions of* M_i, $i \geqslant 5$, *are given by*:

M	$H^2(\mathbf{Z}/p; [M]_p)$	$[M]_p$	
M_5	0	$(p-1)\alpha + \beta$	
$M_6(k)$	$k(\mathbf{Z}/p)$	$k\mathbf{1} + k\alpha + (p-k)\beta$	$0 \leqslant k \leqslant p-1$
$M_7(k)$	$(k+1)(\mathbf{Z}/p)$	$(k+1)\mathbf{1} + k\alpha + (p-k)\beta$	$1 \leqslant k \leqslant p-2$
$M_8(k)$	$k(\mathbf{Z}/p)$	$k\mathbf{1} + (k+1)\alpha + (p-k-1)\beta$	$0 \leqslant k \leqslant p-2$
$M_9(k)$	$(k+1)(\mathbf{Z}/p)$	$(k+1)\mathbf{1} + (k+1)\alpha + (p-k-1)\beta$	$0 \leqslant k \leqslant p-2$

Proof. One begins by computing $H^2(\mathbf{Z}/p, [M]_p)$, from (2.3). We work out an example again with $M = M_6(k)$. We will need these details later. The map

$$p(\mathbf{Z}/p) = H^2(\mathbf{Z}/p, \Lambda_1) \overset{\lambda^k_*}{\to} H^2(\mathbf{Z}/p; \Lambda_1) = p(\mathbf{Z}/p)$$

sends the generator x^j, $0 \leqslant j \leqslant p-1$, from the left-hand side to $x^j(1-x)^k$. The resulting matrix $C_{p,k}$ in $GL_p(\mathbf{Z}/p)$ can be described in the following way. If $p > k$, let $C_{p,k,j}$ denote a column p-vector whose entries are the coefficients of $(1-x)^k$ introduced "cyclically" starting in row j. For example:

$$C_{5,2,4} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$

We define $C_{p,k}$ to be the $p \times p$ matrix whose j^{th} column is $C_{p,k,j}$. So, for example,

$$C_{5,2} = \begin{bmatrix} 1 & 0 & 0 & 1 & -2 \\ -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

It is a consequence of the identity $(1-x)^{k+1} = (1-x)(1-x)^k$ that

$$C_{p,k+1} = C_{p,1}C_{p,k}$$

so we get:

$$C_{p,k} = C^k_{p,1} = \begin{bmatrix} 1 & & & & & -1 \\ -1 & 1 & & & 0 & \\ & -1 & 1 & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & -1 & 1 \end{bmatrix}^k .$$

It is easy to see that $C_{p,1}(p(\mathbf{Z}/p)) = \{\bar{x} \in p(\mathbf{Z}/p) : \sum_{i=0}^{p-1} x_i = 0\}$. Hence rank $(C_{p,k}) = p - k$ and $\dim_{\mathbf{Z}/p} H^2(\mathbf{Z}/p; M_6(k)) = p - (p-k) = k$, as claimed. The other cases are similar.

Now from (2.1) we see that the $\mathbf{Z}/p$-dimension of H^2 is the multiplicity of $\mathbf{1}$ in $[M]_p$. The multiplicities of α and β are then determined by the bottom row of (1.2) restricted to the subgroup of order p.

Recall from Charlap [4] that a cohomology class $\alpha \in H^2(H, M)$ is called *special* if $i * (\alpha) \neq 0$ for the inclusion $i: C \to H$ of any cyclic subgroup. The basic result is (see [4, p. 22]).

(2.6) PROPOSITION. *The extension* Γ *in* (0, 1) *corresponding to* $\alpha \in H^2(H, M)$ *is torsion-free (i.e. the fundamental group of a flat manifold) if and only if* α *is special.*

It remains to determine which indecomposables in (1.3) admit special classes. The result is:

(2.7) PROPOSITION. *There are* $2p - 1$ *genera of indecomposable* $\mathbf{Z}/p^2$*-lattices that admit special classes. They are* M_1, M_4, $M_7(k)$, $1 \leqslant k \leqslant p - 2$, *and* $M_9(k)$, $0 \leqslant k \leqslant p - 2$.

Proof. From (2.2), (2.4) and (2.5) one sees that the given lattices along with $M_6(k)$, $1 \leqslant k \leqslant p - 2$, are the only possibilities. We must determine the restriction map

$$i*(M): H^2(\mathbf{Z}/p^2; M) \to H^2(\mathbf{Z}/p; M)$$

in these cases. Clearly for $M = M_1$, $i*(M)$ is the natural projection and for $M = M_4$, $i*(M)$ is the diagonal embedding.

Now suppose $M = M_6(k)$. We have a commutative diagram of exact sequences from (1.2) and (2.3)

$$\begin{array}{ccccccc} & & H^2(\mathbf{Z}/p^2; \Lambda_1) & \to & H^2(\mathbf{Z}/p^2; M) & \to & 0 \\ & & \downarrow \Delta = i*(\Lambda_1) & & \downarrow i*(M) & & \\ H^2(\mathbf{Z}/p; \varphi_2\Lambda_2) & \overset{\lambda_*^k}{\to} & H^2(\mathbf{Z}/p; \Lambda_1) & \to & H^2(\mathbf{Z}/p; M) & \to & 0 \end{array}$$

where Δ is the diagonal map $\mathbf{Z}_p \to p(\mathbf{Z}/p)$. Hence to eliminate $M_6(k)$, it suffices to show $\mathrm{Im}(\Delta) \subset \mathrm{Im}(\lambda_*^k)$. Let e denote a column p-vector consisting of all 1's, according to the proof of (2.5) we must find an $\bar{x}_k$, $1 \leqslant k \leqslant p-1$ so that $C_{p,k} \cdot \bar{x}_k = C_{p,1}^k \cdot \bar{x}_k = e$. We do this inductively on k. For example, $\bar{x}_1 = \begin{bmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ p \end{bmatrix}$, as can easily be checked. Inductively we define

$$\bar{x}_k(i) = \begin{cases} 1 & i = 1 \\ \bar{x}_k(i-1) + \bar{x}_{k-1}(i) & i > 1 \,. \end{cases}$$

Clearly $C_{p,1} \cdot \bar{x}_k = \bar{x}_{k-1}$, for all coordinates except possibly the first; we must show $\bar{x}_k(p) \equiv 0 \pmod p$. But a comparison of the $\bar{x}_k$'s with Pascal's triangle convinces one that

$$\bar{x}_k(p) = \binom{p-1+k}{p-1} \equiv \binom{k-1}{p-1}\binom{1}{0} \equiv 0 \pmod p,$$

since $k-1 < p-1$.

We leave it for the reader to check that the restriction maps for $M_7(k)$ and $M_9(k)$ are non-trivial.

§ 3. Z/4-MANIFOLDS

In this section, we consider the case $p = 2$. For convenience, we change the notation slightly and write M_7 for $M_6(1)$ and M_i for $M_i(0)$, $i = 6, 8, 9$. According to (2.7), the indecomposable $\mathbf{Z}/4$-lattices that carry special classes are M_1, M_4 and M_9. It is easy to see M_i is faithful if and only if