FLAT MANIFOLDS WITH Z/p^2 HOLONOMY

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By a flat manifold X we will always mean a compact, connected Riemannian manifold of constant curvature zero. Each such space X arises as a quotient E^n/Γ where E^n is n-dimensional Euclidean space and $\Gamma = \pi_1(X)$ is a discrete group of isometries acting freely on E^n, (so X is also called locally Euclidean or a Euclidean space form). The group Γ fits into a short exact sequence:

$$(0.1) \qquad 0 \to M \to \Gamma \to H \to 1$$

where H is the finite holonomy group of X acting faithfully on a free abelian group M of rank n. Furthermore, if N denotes the normalizer of H in $\text{Aut}(M) \cong GL_n(\mathbf{Z})$, then the affine diffeomorphism class of X corresponds precisely to an orbit of N on the "special" classes of $H^2(H, M)$ (see definition preceeding 2.6). Following Charlap [4], we call X an *H-manifold*.

Charlap [4] has given a complete classification of $\mathbf{Z}/p$-manifolds, p a prime number. His results rely on Reiner's description [10] of the integral representation theory of prime order groups.

The success of the classification in this special case depends, in part, on the following result.

THEOREM (A. Jones [9]). *A finite group H admits finitely many isomorphism classes of indecomposable integral representations (i.e. is of finite representation type) if and only if all Sylow p-subgroups of H are cyclic of order p or p^2.*

This result suggests the naturality of generalizing Charlap's classification to $\mathbf{Z}/p^2$-manifolds. In particular, such a classification would contribute to the study of H-manifolds, H cyclic of cube-free order, by using appropriate induction techniques.

The case $H = \mathbf{Z}/p^2$ differs fundamentally from that of $H = \mathbf{Z}/p$ in that the number of genera of indecomposable $\mathbf{Z}/p^2$-lattices is a function of p (in fact, $4p + 1$) while there are 3 genera of indecomposable $\mathbf{Z}/p$-lattices, for any prime p. Furthermore it is no longer the case that only the trivial representation admits non-trivial (and "special") two-dimensional cohomology classes. Eventually we restrict to the case $p = 2$ where there are 9 genera (originally described by Roiter [14] correcting a mistake in Diederichsen [6]) and 3 of these admit special classes. The assumption $p = 2$ also insures that the genera are identical to the isomorphism classes so there are no further invariants to consider. This follows from work of Reiner [13] and the fact that $\mathbf{Z}[e^{2\pi i/m}]$ is a unique factorization domain for $m = 2, 4$.

As the smallest dimension of a $\mathbf{Z}/p^2$-manifold is $p^2 - p + 1$ (this is a special case of results from [8]) only the case $p = 2$ produces flat manifolds of dimension 5, the smallest dimension for which one lacks a complete classification. We show that there are at least 16 5-dimensional flat manifolds with holonomy $\mathbf{Z}/4$ and give a general lower bound for any dimension.

The main ingredient for these results is the work of Heller and Reiner [7] on the integral representation theory of $\mathbf{Z}/p^2$, reviewed in section 1. In section 2 we study the cohomology of the indecomposables, compute their restrictions to the subgroup of order p and identify the "special" classes. Finally in section 3 we restrict to the case $p = 2$ and study the class of $\mathbf{Z}/4$-manifolds.

It is hoped that this example of holonomy classification will succeed in exposing the role played by integral representation theory and cohomology of groups in understanding the structure of flat Riemannian manifolds.

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§ 1. GENERA OF $\mathbf{Z}/p^2$-LATTICES

We begin by briefly reviewing the language and philosophy of the integral representation theory of finite groups (see [5], [11]). We then give Heller and Reiner's description [7] of the genera of $\mathbf{Z}/p^2$ lattices as extensions.

Suppose Λ is a $\mathbf{Z}$-order in a $\mathbf{Q}$-algebra A. A Λ-*lattice* is a left Λ-module that is also a free abelian group of finite rank. The basic problem

of integral representation theory is the classification of such Λ-lattices, Λ fixed. The $\mathbf{Z}$-orders that we will need are group rings of finite groups $\mathbf{Z}G \subset \mathbf{Q}G$ and rings of algebraic integers $\mathcal{O}_K$ in an algebraic number field K. We sometimes refer to a $\mathbf{Z}G$-lattice as a G-lattice.

Let $\mathbf{Z}_p$ (resp. $\mathbf{Q}_p$) denote the p-adic completion of $\mathbf{Z}$ (resp. $\mathbf{Q}$). It is easy to see that $\Lambda_p = \mathbf{Z}_p \otimes \Lambda$ is a $\mathbf{Z}_p$-order in the $\mathbf{Q}_p$-algebra A_p. Furthermore, any Λ-lattice M yields a Λ_p-lattice $M_p = \mathbf{Z}_p \otimes M$. One says that M and M' are *locally isomorphic* (or in the same *genus*) if $M_p \cong M'_p$ as Λ_p-modules for all primes p.

The classification of Λ-lattices is often attacked by a "local-to-global" approach. By this we mean the solution of the following two problems:

1. (local) Determine a complete set of invariants of the genus of a Λ-lattice.

2. (global) Determine a complete set of invariants of the isomorphism class of Λ-lattice within a fixed genus.

This approach has been very successful and the examples below illustrate it.

We introduce some notation. Let ω (resp. ζ) denote a primitive p^2 (resp. p^{th}) root of unity and let $R_1 = \mathbf{Z}[\zeta]$, $R_2 = \mathbf{Z}[\omega]$. We also let $\Lambda_i = \mathbf{Z}[\mathbf{Z}/p^i]$. The classification of lattices over a ring of algebraic integers, or more generally a Dedekind domain is classic, and is a good example of the local-to-global approach.

(1.1) THEOREM (Steinitz). *If R is a Dedekind domain, every R-lattice is a direct sum of non-zero ideals of R. The genus of an R-lattice is determined by the number of non-zero ideals occurring, its rank. The isomorphism class of the R-lattice within the genus is determined by the ideal class of the product of the ideals as an element of the ideal class group of R (the* Steinitz class *of the lattice).*

The classification problem for $\mathbf{Z}/p$-lattices was solved by Diederichsen [6] and Reiner [10]. Again the local-to-global approach is useful.

If $\mathfrak{a}$ denotes a non-zero ideal of R_1 let $E(\mathfrak{a})$ denote the non-split extension of $\mathfrak{a}$ by the trivial lattice $\mathbf{Z}$. The genus of $\mathfrak{a}$ (resp. $E(\mathfrak{a})$) is denoted α (resp. β). Every $\mathbf{Z}/p$-lattice M can be written:

$$M = \mathbf{Z}^{(a)} \oplus \sum_{i=1}^{b} \mathfrak{a}_i \oplus \sum_{i=1}^{c} E(\mathfrak{a}'_i).$$

The genus of the $\mathbf{Z}/p$-lattice M is determined by the multiplicities a, b, c of the three indecomposable genera $\mathbf{1}$, α, β. The isomorphism class of the lattice within its genus is completely determined by the ideal class $\prod_{i=1}^{b} \mathfrak{a}_i \cdot \prod_{i=1}^{c} \mathfrak{a}'_i$ in the ideal class group of R_1.

The solution of the (local) classification problem for $R_2 = \mathbf{Z}[\omega]$ (a Dedekind domain) and $\Lambda_1 = \mathbf{Z}[\mathbf{Z}/p]$ can be combined to classify genera of $\mathbf{Z}/p^2$-lattices. The technique used is essentially homological. If M is a $\mathbf{Z}/p^2$-lattice, we let $L = \{x \in M : (x^p - 1)M = 0\}$. L is a $\mathbf{Z}/p$-lattice and fits into a $\mathbf{Z}/p^2$-exact sequence

$$0 \to L \to M \to N \to 0$$

where N is an R_2-lattice. Hence one is reduced to classifying extensions of R_2-lattices by $\mathbf{Z}/p$-lattices using homological methods. It is not difficult to show (see [13, p. 478]) that $\mathrm{Ext}_2(R_2, L) \equiv L/pL$, where L is an arbitrary $\mathbf{Z}/p$-lattice. In fact, if $\alpha \in L/pL$ then the corresponding extension is given by the pushout diagram:

$$
\begin{array}{ccccccccc}
0 & \to & \varphi_2\Lambda_2 & \to & \Lambda_2 & \to & R_2 & \to & 0 \\
 & & {\scriptstyle\alpha}\downarrow & & \downarrow & & \| & & \\
0 & \to & L & \to & M & \to & R_2 & \to & 0
\end{array}
$$

(1.2)

where φ_2 is the cyclotomic polynomial $1 + x^p + x^{2p} + \ldots + x^{(p-1)p}$ and the map α is (by abuse of language) the map that sends φ_2 to α. We write (L, α) for this extension. The final result is:

(1.3) THEOREM (Heller and Reiner [7]). *There are $4p + 1$ genera of indecomposable $\mathbf{Z}/p^2$-lattices given by:*

$$M_1 = \mathbf{Z},$$
$$M_2 = R_1,$$
$$M_3 = R_2,$$
$$M_4 = \Lambda_1,$$
$$M_5 = (\mathbf{Z}, 1),$$

$M_6(k) = (\Lambda_1, \lambda^k),$	$0 \leqslant k \leqslant p - 1,$
$M_7(k) = (Z \oplus \Lambda_1, 1 \oplus \lambda^k),$	$1 \leqslant k \leqslant p - 2,$
$M_8(k) = (R_1, \lambda^k),$	$0 \leqslant k \leqslant p - 2,$
$M_9(k) = (Z \oplus R_1, 1 \oplus \lambda^k),$	$0 \leqslant k \leqslant p - 2,$

where $\lambda = (1 - x)$, $\Lambda_1 \cong \mathbf{Z}[x]/(x^p - 1)$ and we view R_1 as a quotient of Λ_1.

The splintering of these genera into isomorphism classes has been analyzed by Reiner [13]. One can, of course, replace R_1, R_2 by ideal classes in these rings and Λ_1 by $E(\mathfrak{a})$ (cf. section 1) where $\mathfrak{a}$ is an ideal class in R_1. There is an additional invariant lying in a quotient of the group of units of a certain finite ring and, if $p \equiv 1$ (mod 4) a certain quadratic residue character mod p can also appear as an invariant. The precise result is Theorem 7.3 of [13]. We will require only the observation [13, p. 494] that if $p = 2, 3$ there are no further invariants, i.e. each genus of an indecomposable is a single isomorphism class. In the case $p = 5$ already, although the class number of $\mathbf{Q}(e^{2\pi i/m})$ is one for $m = 5, 25$, the 21 genera of indecomposables split up into 40 isomorphism classes. Hence already the further isomorphism invariants mentioned above exert an influence.

§ 2. Cohomology, restrictions and special classes

If H is a finite group, M an H-lattice then: $H^i(H, M) \cong \oplus_p H^i(H, M_p)$, where p ranges over the primes dividing the order of H [3, p. 84]. Hence if M and M' are locally isomorphic, $H^i(H, M) \cong H^i(H, M')$; so the cohomology of an H-lattice depends only on its genus.

We recall the cohomology of a cyclic group $\mathbf{Z}/n = \langle\sigma\rangle$ [3, p. 58]. We write $N = 1 + \sigma \dots + \sigma^{n-1}$ and $D = 1 - \sigma$. If M is a $\mathbf{Z}/n$-module, then

$$H^0(\mathbf{Z}/n, M) = M^\sigma$$

$$H^{2i-1}(\mathbf{Z}/n, M) = {}_N M/D \cdot M$$

$$H^{2i}(\mathbf{Z}/n, M) = M^\sigma/N \cdot M$$

for all $i \geqslant 1$, where M^σ denotes σ-invariants and ${}_N M = \{x \in M : Nx = 0\}$. From these remarks it is easy to compute the cohomology of the indecomposable $\mathbf{Z}/p$-lattices described in section 1.

(2.1) Proposition. *The following table describes the cohomology of the indecomposable $\mathbf{Z}/p$-lattices:*

M	rank	H^0	H^1	H^2
$\mathbf{1}$	1	$\mathbf{Z}$	0	$\mathbf{Z}/p$
α	$p - 1$	0	$\mathbf{Z}/p$	0
β	p	$\mathbf{Z}$	0	0

Similarly one can easily compute the cohomology and restriction of the first four $\mathbf{Z}/p^2$-lattices of (1.3).

(2.2) PROPOSITION. *If* $[M]_p$ *denotes the restriction of the* $\mathbf{Z}/p^2$-*lattice* M *to the subgroup of order* p, *we have the following table.*

M	rank	H^0	H^1	H^2	$[M]_p$
$M_1 = \mathbf{1}$	1	$\mathbf{Z}$	0	$\mathbf{Z}/p^2$	$\mathbf{1}$
$M_2 = R_1$	$p-1$	0	$\mathbf{Z}/p$	0	$(p-1)\mathbf{1}$
$M_3 = R_2$	$p^2 - p$	0	$\mathbf{Z}/p$	0	$p\alpha$
$M_4 = \Lambda_1$	p	$\mathbf{Z}$	0	$\mathbf{Z}/p$	$p\mathbf{1}$

Furthermore, $\Lambda_2 = \mathbf{Z}[\mathbf{Z}/p^2]$, the regular representation of $\mathbf{Z}/p^2$, satisfies $H^0 = \mathbf{Z}$, $H^1 = 0 = H^2$ and $[\Lambda_2]_\beta = p\beta$.

Proof. It suffices to observe that $M_2 = p^*(\alpha)$ and $M_4 = p^*(\beta)$, where $p: \mathbf{Z}/p^2 \to \mathbf{Z}/p$ is the natural projection, and M_3 fits into a short exact sequence:

$$0 \to M_4 \to \Lambda_2 \to M_3 \to 0 .$$

The last remark follows from the freeness of Λ_2.

To complete the table for the modules $M_i, i \geqslant 5$, we have the following lemma:

(2.3) LEMMA. *If* L *is a* $\mathbf{Z}/p$-*lattice,* $\alpha \in L$, *then the extension* $M = (L, \alpha)$ *defined by (1.2) satisfies:*

$$H^2(C; M) \cong \operatorname{coker}\left(x_*: H^2(C; \varphi_2 \Lambda_2) \to H^2(C; L)\right) .$$

where C *is either* $\mathbf{Z}/p^2$ *or* $\mathbf{Z}/p$.

Proof. The diagram (1.2) induces

$$
\begin{array}{ccccc}
\to & H^1(C, R_2) & \overset{\delta}{\to} & H^2(C, \varphi_2\Lambda_2) & \to & 0 \\
 & \downarrow & & \downarrow x_* & & \\
\to & H^1(C, R_2) & \underset{\delta}{\to} & H^2(C, L) & \to & H^2(C, M) & \to & 0
\end{array}
$$

where the zeros follow from (2.2). An easy diagram-chase completes the proof.

(2.4) PROPOSITION. *The following table describes the cohomology of the indecomposable $\mathbb{Z}/p^2$-lattices $M_i, i \geqslant 5$:*

M	rank	H^0	H^1	H^2
M_5	$p^2 - p + 1$	$\mathbb{Z}$	0	$\mathbb{Z}/p$
$M_6(0)$	p^2	$\mathbb{Z}$	0	0
$M_6(k)$	p^2	$\mathbb{Z}$	$\mathbb{Z}/p$	$\mathbb{Z}/p$
$M_7(k)$	$p^2 + 1$	$\mathbb{Z} \oplus \mathbb{Z}$	0	$\mathbb{Z}/p \oplus \mathbb{Z}/p$
$M_8(0)$	$p^2 - 1$	0	$\mathbb{Z}/p^2$	0
$M_8(k)$	$p^2 - 1$	0	$\mathbb{Z}/p \oplus \mathbb{Z}/p$	0
$M_9(k)$	p^2	$\mathbb{Z}$	$\mathbb{Z}/p$	$\mathbb{Z}/p$

Proof. Since

$$H^0(\mathbb{Z}/p^2 ; R_2) = 0, \quad H^0(\mathbb{Z}/p^2 ; (L, x)) \cong H^0(\mathbb{Z}/p^2 ; L)$$

and these can be read off from (2.2). The groups H^2 are computed by (2.3). We work out one example in detail. Consider $M_6(k), 0 \leqslant k \leqslant p - 1$, so that $L = \Lambda_1$. If we identify $\varphi_2 \Lambda_2$ with Λ_1 then the generator:

$$1 + x + \ldots + x^{p-1} \in H^2(\mathbb{Z}/p^2, \varphi_2 \Lambda_2)$$

is sent by $\lambda_*^k, 1 \leqslant k \leqslant p - 1$, to

$$(1-x)^k(1+x+\ldots+x^{p-1}) = (1-x)^{k-1} \cdot 0 = 0$$

in $H^2(\mathbb{Z}/p^2 ; \Lambda_1)$. If $k = 0$, then the map is an isomorphism. Hence $H^2(\mathbb{Z}/p^2 ; M_6(0)) = 0$ and $H^2(\mathbb{Z}/p^2, M_6(k)) = \mathbb{Z}/p, k \geqslant 1$.

The groups $H^1(M_i)$ can be read off the long exact cohomology sequence of the bottom row of (1.2).

Remark. It follows from (2.2) that $M_6(0)$ is the genus of the regular representation.

We now record the restrictions of the modules M_i to the subgroup of order p.

(2.5) PROPOSITION. *The* $\mathbf{Z}/p$-*cohomology and the restrictions of* $M_i, i \geqslant 5,$ *are given by*:

M	$H^2(\mathbf{Z}/p; [M]_p)$	$[M]_p$	
M_5	0	$(p-1)\alpha + \beta$	
$M_6(k)$	$k(\mathbf{Z}/p)$	$k\mathbf{1} + k\alpha + (p-k)\beta$	$0 \leqslant k \leqslant p-1$
$M_7(k)$	$(k+1)(\mathbf{Z}/p)$	$(k+1)\mathbf{1} + k\alpha + (p-k)\beta$	$1 \leqslant k \leqslant p-2$
$M_8(k)$	$k(\mathbf{Z}/p)$	$k\mathbf{1} + (k+1)\alpha + (p-k-1)\beta$	$0 \leqslant k \leqslant p-2$
$M_9(k)$	$(k+1)(\mathbf{Z}/p)$	$(k+1)\mathbf{1} + (k+1)\alpha + (p-k-1)\beta$	$0 \leqslant k \leqslant p-2$

Proof. One begins by computing $H^2(\mathbf{Z}/p, [M]_p)$, from (2.3). We work out an example again with $M = M_6(k)$. We will need these details later. The map

$$p(\mathbf{Z}/p) = H^2(\mathbf{Z}/p, \Lambda_1) \overset{\lambda^k_*}{\to} H^2(\mathbf{Z}/p; \Lambda_1) = p(\mathbf{Z}/p)$$

sends the generator x^j, $0 \leqslant j \leqslant p - 1$, from the left-hand side to $x^j(1-x)^k$. The resulting matrix $C_{p,k}$ in $GL_p(\mathbf{Z}/p)$ can be described in the following way. If $p > k$, let $C_{p,k,j}$ denote a column p-vector whose entries are the coefficients of $(1-x)^k$ introduced "cyclically" starting in row j. For example:

$$C_{5,2,4} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -2 \end{bmatrix}$$ We define $C_{p,k}$ to be the $p \times p$

matrix whose j^{th} column is $C_{p,k,j}$. So, for example,

$$C_{5,2} = \begin{bmatrix} 1 & 0 & 0 & 1 & -2 \\ -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

It is a consequence of the identity $(1-x)^{k+1} = (1-x)(1-x)^k$ that

$$C_{p,k+1} = C_{p,1}C_{p,k}$$

so we get:

$$C_{p,k} = C_{p,1}^k = \begin{bmatrix} 1 & & & & & -1 \\ -1 & 1 & & & & \\ & & & & & 0 \\ & -1 & 1 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & -1 & 1 \end{bmatrix}^k .$$

It is easy to see that $C_{p,1}(p(\mathbf{Z}/p)) = \{\bar{x} \in p(\mathbf{Z}/p): \sum_{i=0}^{p-1} x_i = 0\}$. Hence rank $(C_{p,k}) = p - k$ and $\dim_{\mathbf{Z}/p} H^2(\mathbf{Z}/p; M_6(k)) = p - (p-k) = k$, as claimed. The other cases are similar.

Now from (2.1) we see that the $\mathbf{Z}/p$-dimension of H^2 is the multiplicity of 1 in $[M]_p$. The multiplicities of α and β are then determined by the bottom row of (1.2) restricted to the subgroup of order p.

Recall from Charlap [4] that a cohomology class $\alpha \in H^2(H, M)$ is called *special* if $i*(\alpha) \neq 0$ for the inclusion $i: C \to H$ of any cyclic subgroup. The basic result is (see [4, p. 22]).

(2.6) PROPOSITION. *The extension* Γ *in* (0, 1) *corresponding to* $\alpha \in H^2(H, M)$ *is torsion-free (i.e. the fundamental group of a flat manifold) if and only if* α *is special.*

It remains to determine which indecomposables in (1.3) admit special classes. The result is:

(2.7) PROPOSITION. *There are* $2p - 1$ *genera of indecomposable* $\mathbf{Z}/p^2$- *lattices that admit special classes. They are* M_1, M_4, $M_7(k)$, $1 \leqslant k \leqslant p - 2$, *and* $M_9(k)$, $0 \leqslant k \leqslant p - 2$.

Proof. From (2.2), (2.4) and (2.5) one sees that the given lattices along with $M_6(k)$, $1 \leqslant k \leqslant p - 2$, are the only possibilities. We must determine the restriction map

$$i*(M): H^2(\mathbf{Z}/p^2; M) \to H^2(\mathbf{Z}/p; M)$$

in these cases. Clearly for $M = M_1$, $i*(M)$ is the natural projection and for $M = M_4$, $i*(M)$ is the diagonal embedding.

Now suppose $M = M_6(k)$. We have a commutative diagram of exact sequences from (1.2) and (2.3)

$$H^2(\mathbf{Z}/p^2; \Lambda_1) \quad\to\quad H^2(\mathbf{Z}/p^2; M) \quad\to\quad 0$$

$$\downarrow \scriptstyle \Delta = i*(\Lambda_1) \qquad\qquad\qquad \downarrow \scriptstyle i*(M)$$

$$H^2(\mathbf{Z}/p; \varphi_2\Lambda_2) \quad\overset{\lambda^k_*}{\to}\quad H^2(\mathbf{Z}/p; \Lambda_1) \quad\to\quad H^2(\mathbf{Z}/p; M) \quad\to\quad 0$$

where Δ is the diagonal map $\mathbf{Z}_p \to p(\mathbf{Z}/p)$. Hence to eliminate $M_6(k)$, it suffices to show $\mathrm{Im}(\Delta) \subset \mathrm{Im}(\lambda^k_*)$. Let e denote a column p-vector consisting of all 1's, according to the proof of (2.5) we must find an $\overline{x}_k$, $1 \leqslant k \leqslant p - 1$ so that $C_{p,k} \cdot \overline{x}_k = C^k_{p,1} \cdot \overline{x}_k = e$. We do this inductively

on k. For example, $\overline{x}_1 = \begin{bmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ p \end{bmatrix}$, as can easily be checked. Inductively we

define

$$\overline{x}_k(i) = \begin{cases} 1 & i = 1 \\ \\ \overline{x}_k(i-1) + \overline{x}_{k-1}(i) & i > 1. \end{cases}$$

Clearly $C_{p,1} \cdot \overline{x}_k = \overline{x}_{k-1}$, for all coordinates except possibly the first; we must show $\overline{x}_k(p) \equiv 0 \pmod p$. But a comparison of the $\overline{x}_k$'s with Pascal's triangle convinces one that

$$\overline{x}_k(p) = \binom{p-1+k}{p-1} \equiv \binom{k-1}{p-1}\binom{1}{0} \equiv 0 \pmod p,$$

since $k - 1 < p - 1$.

We leave it for the reader to check that the restriction maps for $M_7(k)$ and $M_9(k)$ are non-trivial.

§ 3.   Z/4-MANIFOLDS

In this section, we consider the case $p = 2$. For convenience, we change the notation slightly and write M_7 for $M_6(1)$ and M_i for $M_i(0)$, $i = 6, 8, 9$. According to (2.7), the indecomposable Z/4-lattices that carry special classes are M_1, M_4 and M_9. It is easy to see M_i is faithful if and only if

$i = 3, 5, 6, 7, 8, 9$. Hence if $M = \sum m_i M_i$ is an arbitrary $\mathbf{Z}/4$-lattice then M is a faithful representation carrying a special class if and only if the multiplicities m_i satisfy the inequalities:

$$m_1 + m_4 + m_9 > 0$$

(3.0)

$$m_3 + m_5 + m_6 + m_7 + m_8 + m_9 > 0.$$

Since the multiplicities are a complete set of isomorphism invariants in the case $p = 2$ (see section 1) one can use the conditions (3.0) to show:

(3.1) THEOREM. *If* $L_n(m)$ *denotes the number of isomorphism classes of n-dimensional* $\mathbf{Z}/m$-*lattices that carry special classes, then*:

$$L_n(4) = \sum_{j=2}^{n-1} \left(a_j - \left[\frac{j}{2}\right] - 1 \right) + \sum_{j=[n]_2+2}^{n-2} (a_j - a_{j-1} - 1)$$

$$+ \sum_{j=[n]_4}^{n-4} (a_j - a_{j-2} - a_{j-4} + a_{j-6})$$

where $[k]_p$ denotes the reduction of k modulo p, $[k]$ denote the largest integer $\leqslant k$ and the a_j's are given by

$$P(t) = \sum_{j=0}^{\infty} a_j t^j = \frac{1}{(1-t)(1-t^2)^2(1-t^3)^2(1-t^4)^3}$$

In particular, the number of n-dimensional $\mathbf{Z}/4$-manifolds is at least $L_n(4)$.

Proof. If $Q(t)$ is a power series, let $\mathrm{coef}(n, Q(t))$ denote the coefficient of t^n in $Q(t)$. The number $L_n(4)$ counts the number of ways of writing

$$n = m_1 + m_2 + 2(m_3+m_4) + 3(m_5+m_8) + 4(m_6+m_7+m_9)$$

where the m_i's satisfy (3.0). If $m_1 > 0$ there is a contribution:

$$\sum_{m_1=1}^{n-2} \mathrm{coef}(n-m_1, P(t)) - \left(\left[\frac{n-m_1}{2}\right] + 1 \right)$$

where $\left[\dfrac{n-m_1}{2}\right] + 1$ is the number of ways of expressing $n - m_1$ as a combination of 1's (M_2) and 2's (M_4) (not permitted by (3.0)). Reindexing gives the first term for $L_n(4)$.

Similarly, if $m_1 = 0$, $m_4 > 0$ there is a contribution:

$$\sum_{m_4} \mathrm{coef}(t^{n-2m_4}(1-t)P(t)) - 1$$

where 1 is subtracted to omit choosing m_2 alone. Finally, if $m_1 = m_4 = 0$, we have:

$$\sum_{m_9} \text{coef}(t^{n-4m_9}, (1-t^2)(1-t^4)P(t)) \, .$$

The coefficients of $(1-t)P(t)$ and $(1-t^2-t^4-t^6)P(t)$ are easily expressible in terms of the a_j's and the result follows.

Remark. In order for a Z/p-lattice to carry a special class, the multiplicity of the trivial representation must be non-zero. Topologically this is reflected in the fact that a Z/p-manifold fibers over a circle. This is already false for a 4-dimensional $Z/4$-manifold as the following example shows.

Example. $L_4(4) = 6$. The multiplicities of the indecomposables in these 4-dimensional $Z/4$-lattices are given by:

notation of [2]	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9
07/02/02	1				1				
12/01/04	1							1	
12/01/02	1	1	1						
07/02/01	2			1					
12/01/03			1	1					
12/01/06									1

where the first column gives the label of these "Z-classes" from the table of the four-dimensional crystallographic groups in [2]. In fact, as these tables indicate, there is precisely one $Z/4$-manifold corresponding to each $Z/4$-lattice, hence there are exactly 6 4-dimensional $Z/4$-manifolds.

Remark. Recall that if $p < 23$, the field $Q(e^{2\pi i/p})$ has class number one. This fact, along with the work of Charlap [4], shows that the number of n-dimensional Z/p-manifolds is exactly $L_n(p)$, $p < 23$. This number is readily computable, as Charlap [4, p. 30] remarks, and the precise formula is:

$$(3.2) \qquad L_n(p) = \sum_{j=p-1}^{n-1} \left(\left[\frac{j}{p-1} \right] - <\frac{j}{p}> + 1 \right)$$

where $<k>$ denotes the smallest integer $\geq k$. In particular, $L_p(p) = 1$, $L_n(p) = 0$, $p > n$, and when $p = 2$

(3.3) $$L_n(2) = \left(\frac{n}{2}\right)^2 + \left(\frac{n}{2}\right) - 1 + \frac{[n]_2}{4}.$$

using the notation of (3.1).

One can easily construct the following table of values of $L_n(p)$:

n	2	3	4	5	6
p					
2	1	3	5	8	11
3		1	2	3	4
5				1	2

Hence 14 of the 74 4-dimensional flat manifolds have cyclic holonomy $\leqslant 5$. (Furthermore, 26 have holonomy the Klein 4-group.) We describe analogous facts in dimension 5 below.

We let $SH^2(H, M)$ denote the set of special classes in $H^2(H, M)$. If H is a cyclic p-group and $i: \mathbf{Z}_p \hookrightarrow H$ is the inclusion of the subgroup of order p, then

$$SH^2(H, M) = H^2(H, M) - \ker(i*).$$

If N (resp. Z) denotes the normalizer (resp. the centralizer) of H in $\mathrm{Aut}(M)$, there is an exact sequence (see [15, p. 50])

$$0 \to Z \to N \to \mathrm{Aut}(H).$$

We conjecture:

Conjecture. If $\mathbf{Z}[e^{2\pi i/p^k}]$ is a unique factorization domain for, $1 \leqslant k \leqslant n$, then N acts transitively on $SH^2(\mathbf{Z}/p^n; M)$ for any H-lattice M.

The case $n = 1$ of the Conjecture follows from Charlap [4]. Class number tables shows that the $n = 2$ case applies to $p = 2, 3, 5$, the $n = 3$ case to $2, 3$ and the $n = 4$ case to $p = 2$. This conjecture implies that the lower bound of (3.1) is exact.

We mention that the multiplicities of the indecomposables in the 5-dimensional $\mathbf{Z}/4$-lattices that admit special classes are given by:

	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9
	1								1
	1						1		
*						1			
	1	1			1				
*	1	1						1	
	1		1	1					
*	1		2						
*	1	2	1						
*	3							1	
	2					1			
*	2	1	1						
	3		1						
*		1	1	1					
				1	1				
*				1				1	
*		1							1

Those lattices that are starred clearly satisfy the Conjecture.

REFERENCES

[1] BERMAN, S. and P. GUDIKOV. Indecomposable representations of finite groups over the ring of p-adic integers. *Amer. Math. Soc. Trans., Series 2, vol. 50* (1966), 77-113.

[2] BROWN, H., R. BÜLOW, J. NEUBÜSER, H. WONDRATSCHEK and H. ZASSENHAUS. *Crystallographic groups of four-dimensional space.* John Wiley, New York, 1978.

[3] BROWN, K. *Cohomology of Groups.* Springer Verlag, New York, 1983.

[4] CHARLAP, L. Compact flat Riemannian manifolds I. *Ann. Math. 81* (1965), 15-30.

[5] CURTIS, C. W. and I. REINER. *Representation theory of finite groups and associative algebras.* Interscience, New York, 1966.

[6] DIEDERICHSEN, F. Über die Ausreduktion ganzzahliger Guppendarstellungen bei arithmetischer Äquivalenz. *Abh. Hans. Univ. 13* (1938), 357-412.

[7] HELLER, A. and I. REINER. Representations of cyclic groups in rings of integers, I, II. *Ann. Math. (2) 76* (1962), 73-92 and 77 (1963), 318-328.

[8] HILLER, H. Minimal dimension of flat manifolds with abelian holonomy. *To appear.*

[9] JONES, A. Groups with a finite number of indecomposable integral representations. *Mich. Math. J. 10* (1963), 257-261.

[10] REINER, I. Integral representations of cyclic groups of prime order. *Proc. Amer. Math. Soc. 8* (1957), 142-146.

[11] —— Integral representations of cyclic groups of order p^2. *Proc. Amer. Math. Soc. 58* (1976), 8-12.

[12] —— A survey of integral representation theory. *Bull. A.M.S. 76* (1970), 159-227.

[13] —— Invariants of integral representations. *Proc. J. Math. 78* (1978), 467-501.

[14] ROITER, A. V. On the representations of the cyclic group of fourth order by integral matrices. *Vestnik Leningrad Univ. 15 No. 19* (1960), 65-74 (Russian).

[15] SUZUKI, M. *Group theory*. Springer Verlag, New York, 1982.

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