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# REPRESENTATION OF REAL NUMBERS BY MEANS OF FIBONACCI NUMBERS

## by Paulo RIBENBOIM

Dedicated to Professor D.H. Lehmer at the occasion of his eightieth birthday.

The purpose of this note is to derive a new representation of positive real numbers as sums of series involving Fibonacci numbers. This will be an easy application of an old result of Kakeya [4]. The paper concludes with a result of Landau [5], relating the sum  $\sum_{n=1}^{\infty} \frac{1}{F_n}$  with values of theta series; we believe it worthwhile to unearth Landau's result, which is now rather inaccessible.

1. Let  $(s_i)_{i \ge 1}$  be a sequence of positive real numbers, such that  $s_1 > s_2 > s_3 > \dots$  and  $\lim_{i \to \infty} s_i = 0$ . Let  $S = \sum_{i=1}^{\infty} s_i \le \infty$ .

We say that x > 0 is representable by the sequence  $(s_i)_{i \ge 1}$  if  $x = \sum_{j=1}^{\infty} s_{i_j}$  (with  $i_1 < i_2 < i_3 < \ldots$ ). Then necessarily  $x \le S$ .

The first result is due to Kakeya; for the sake of completeness, we give a proof:

Proposition 1. The following conditions are equivalent:

- 1) Every  $x, 0 < x \leq S$ , is representable by the sequence  $(s_i)_{i \geq 1}, x = \sum_{j=1}^{\infty} s_{i_j}$ , where  $i_1$  is the smallest index such that  $s_{i_1} < x$ .
- 2) Every x, 0 < x < S is representable by the sequence  $(s_i)_{i \ge 1}$ .
- 3) For every  $n \ge 1$ ,  $s_n \le \sum_{i=n+1}^{\infty} s_i$ .

*Proof.*  $1 \rightarrow 2$ . This is trivial.

 $2 \to 3$ . If there exists  $n \ge 1$  such that  $s_n > \sum_{i=n+1}^{\infty} s_i$ , let x be such that  $s_n > x > \sum_{i=n+1}^{\infty} s_i$ . By hypothesis,  $x = \sum_{j=1}^{\infty} s_{i_j}$  with  $i_1 < i_2 < \dots$  Since  $s_n > x > s_{i_1}$ , then  $n < i_1$ , hence  $x = \sum_{j=1}^{\infty} s_{i_j} \le \sum_{k=n+1}^{\infty} s_k$ , which is absurd.

 $3 \to 1$ . Since  $\lim_{i \to \infty} s_i = 0$ , there exists the smallest index  $i_1$  such that  $s_{i_1} < x$ .

Similarly, there exists the smallest index  $i_2$  such that  $i_1 < i_2$  and  $s_{i_2} < x - s_{i_1}$ .

More generally, for every  $n \ge 1$  we define  $i_n$  to be the smallest index such that  $i_{n-1} < i_n$  and  $s_{i_n} < x - \sum_{i=1}^{n-1} s_{i_i}$ .

Then 
$$x \ge \sum_{j=1}^{\infty} s_{i_j}$$
. Suppose that  $x > \sum_{j=1}^{\infty} s_{i_j}$ .

We note that there exists N such that if  $m \ge N$  then  $s_{i_m} < x - \sum_{j=1}^m s_{i_j}$ . Otherwise, there exist infinitely many indices  $n_1 < n_2 < n_3 < \dots$  such that  $s_{i_{n_k}} \ge x - \sum_{j=1}^{n_k} s_{i_j}$ . At the limit, we have  $0 = \lim_{k \to \infty} s_{i_{n_k}} \ge x - \sum_{j=1}^{\infty} s_{i_j} > 0$ , and this is a contradiction.

We choose N minimal with above property.

We show: for every  $m \ge N$ ,  $i_m + 1 = i_{m+1}$ . In fact

$$s_{i_m+1} < s_{i_m} < x - \sum_{j=1}^m s_{i_j},$$

so by definition of the sequence of indices,  $i_m + 1 = i_{m+1}$ . Therefore the following sets coincide:  $\{i_N, i_N + 1, i_N + 2, \ldots\} = \{i_N, i_{N+1}, i_{N+2}, \ldots\}$ .

Next we show that  $i_N = 1$ . If  $i_N > 1$  we consider the index  $i_N - 1$ , and by hypothesis (3):

$$s_{i_N-1} \leqslant \sum_{k=i_N}^{\infty} s_k = \sum_{j=N}^{\infty} s_{i_j} < x - \sum_{j=1}^{N-1} s_{i_j}.$$

We have  $i_{N-1} \leq i_N - 1 < i_N$ . If  $i_{N-1} < i_N - 1$  this is impossible, because  $i_N$  was defined to be the smallest index such that  $i_{N-1} < i_N$  and  $s_{i_N} < x - \sum_{j=1}^{N-1} s_{i_j}$ . Thus  $i_{N-1} = i_N - 1$ , that is  $s_{i_{N-1}} < x - \sum_{j=1}^{N-1} s_{i_j}$  and this is against the choice of N as minimal with the property indicated.

Thus 
$$i_N = 1$$
 and  $x > \sum_{j=1}^{\infty} s_{i_j} = \sum_{i=1}^{\infty} s_i = S$ , against the hypothesis.  $\square$ 

We remark now that if the above conditions are satisfied for the sequence  $(s_i)_{i \ge 1}$ , if  $m \ge 0$  then every x,  $0 < x < S' = \sum_{i=m+1}^{\infty} s_i$  is representable by the sequence  $(s_i)_{i \ge m+1}$  with  $i_1$  the smallest index such that  $m+1 \le i_1$  and  $s_{i_1} < x$ .

Indeed, condition (3) holds for  $(s_i)_{i \ge 1}$  hence also for  $(s_i)_{i \ge m+1}$ . Since 0 < x < S', the remark follows from the proposition.

Proposition 1 has been generalized (see for example Fridy [3]). Now we consider the question of unique representation (this was generalized by Brown in [1]).

Proposition 2. With above notations, the following conditions are equivalent:

- 2') Every x, 0 < x < S, has a unique representation  $x = \sum_{i=1}^{\infty} s_{i_i}$ .
- 3') For every  $n \ge 1$ ,  $s_n = \sum_{i=n+1}^{\infty} s_i$ .
- 4) For every  $n \ge 1$ ,  $s_n = \frac{1}{2^{n-1}} s_1$  (hence  $S = 2s_1$ ).

*Proof.*  $2' \to 3'$ . Suppose there exists  $n \ge 1$  such that  $s_n \ne \sum_{i=n+1}^{\infty} s_i$ . Since (2') implies (2) hence also (3) then  $s_n < \sum_{i=n+1}^{\infty} s_i$ . Let x be such that  $s_n < x < \sum_{i=n+1}^{\infty} s_i$ . By the above remark, x is representable by the sequence  $\{s_i\}_{i \ge n+1}$ , that is  $x = \sum_{\substack{j=1 \ k_j \ge n+1}}^{\infty} s_{k_j}$ . On the other hand, (2') implies (2), hence

also (1) and x has a representation  $x = \sum_{j=1}^{\infty} s_{i_j}$ , where  $i_1$  is the smallest index such that  $s_{i_1} < x$ . From  $s_n < x$  it follows that  $i_1 \le n$  and so x would have two distinct representations, against the hypothesis.

 $3' \to 4'$ . We have  $s_n = s_{n+1} + \sum_{i=n+2}^{\infty} s_i = 2s_{n+1}$  for every  $n \ge 1$  hence  $s_n = \frac{1}{2^{n-1}} s_1$  for every  $n \ge 1$ .

 $4' \rightarrow 2'$ . Suppose that there exists x, 0 < x < S having two distinct representations

$$x = \sum_{j=1}^{\infty} s_{i_j} = \sum_{j=1}^{\infty} s_{k_j}.$$

Let  $j_0$  be the smallest index such that  $i_{j_0} \neq k_{j_0}$ , say  $i_{j_0} < k_{j_0}$ . Then  $\sum_{j=j_0}^{\infty} s_{i_j} = \sum_{j=j_0}^{\infty} s_{k_j} \leqslant \sum_{n=i_{j_0}+1}^{\infty} s_n.$ 

By hypothesis, after dividing by  $s_1$ , we have

$$\sum_{n=i_{j_0}}^{\infty} \frac{1}{2^n} \geqslant \sum_{j=j_0}^{\infty} 2^{1-k_j} = \sum_{j=j_0}^{\infty} 2^{1-i_j}$$

$$= 2^{1-i_{j_0}} + \sum_{j=j_0+1}^{\infty} 2^{1-i_j} = \sum_{n=i_{j_0}}^{\infty} 2^{-n} + \sum_{j=j_0+1}^{\infty} 2^{1-i_j}$$

hence  $\sum_{i=j_0+1}^{\infty} 2^{1-i_j} \leq 0$ , which is impossible.

For practical applications, we note:

If  $s_n \leq 2s_{n+1}$  for every  $n \geq 1$  then condition (3) is satisfied. Indeed

$$\sum_{i=n+1}^{\infty} s_i \leqslant 2 \sum_{i=n+1}^{\infty} s_{i+1} = 2 \sum_{i=n+2}^{\infty} s_i, \quad \text{hence} \quad s_{n+1} \leqslant \sum_{i=n+2}^{\infty} s_i$$

and  $s_n \leqslant 2 s_{n+1} \leqslant \sum_{i=n+1}^{\infty} s_i$ .

2. Now we give various ways of representing real numbers.

First, the dyadic representation, which may of course be easily obtained directly:

COROLLARY 1. Every real number x, 0 < x < 1, may be written uniquely in the form  $x = \sum_{j=1}^{\infty} \frac{1}{2^{n_j}}$  (with  $1 \le n_1 < n_2 < n_3 < \dots$ ).

*Proof.* This has been shown in proposition 2, taking 
$$s_1 = \frac{1}{2}$$
.

COROLLARY 2. Every positive real number x may be written in the form  $x = \sum_{j=1}^{\infty} \frac{1}{n_j}$  (with  $n_1 < n_2 < n_3 < \dots$ ).

*Proof.* We consider the sequence  $\left(\frac{1}{n}\right)_{n\geq 1}$ , which is decreasing with limit equal to zero, and we note that  $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$  and  $\frac{1}{n} \leq \frac{2}{n+1}$  for every  $n \geq 1$ .

Thus by Kakeya's theorem and the above remark, every x > 0 is representable as indicated.

COROLLARY 3. Every positive real number x may be written as  $x = \sum_{j=1}^{\infty} \frac{1}{p_{i_j}}$  (where  $p_1 < p_2 < p_3 < \ldots$  is the increasing sequence of prime numbers).

*Proof.* We consider the sequence  $\left(\frac{1}{p_i}\right)_{i \ge 1}$ , which is decreasing with limit equal to zero. As Euler proved  $\sum_{i=1}^{\infty} \frac{1}{p_i} = \infty$ . By Tschebycheff's theorem (proof of Bertrand's "postulate") there is a prime in each interval (n, 2n); thus  $p_{i+1} < 2$   $p_i$  and  $\frac{1}{p_i} < \frac{2}{p_{i+1}}$  for every  $i \ge 1$ . By Kakeya's theorem and the above remark, every x > 0 is representable as indicated.

3. Now we shall represent real numbers by means of Fibonacci numbers and we begin giving some properties of these numbers.

The Fibonacci numbers are:  $F_1 = F_2 = 1$  and for every  $n \ge 3$ ,  $F_n$  is defined by the recurrence relation  $F_n = F_{n-1} + F_{n-2}$ .

Thus the sequence of Fibonacci numbers is

In the following proposition, we give a closed form expression for the Fibonacci numbers; this is due to Binet (1843).

Let  $\alpha = \frac{\sqrt{5} + 1}{2}$  (the golden number) and  $\beta = -\frac{\sqrt{5} - 1}{2}$ , so  $\alpha + \beta = 1$ ,  $\alpha\beta = -1$ , thus  $\alpha$ ,  $\beta$  are the roots of  $X^2 - X - 1 = 0$  and  $-1 < \beta < 0 < 1 < \alpha$ .

We have

LEMMA 1. For every 
$$n \geqslant 1$$
,  $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$  and  $\frac{\alpha^{n-1}}{\sqrt{5}} < F_n < \frac{\alpha^{n+1}}{\sqrt{5}}$ .

*Proof.* We consider the sequence of numbers  $G_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$  for  $n \ge 1$ . Then  $G_1 = G_2 = 1$ ; moreover

$$G_{n-1} + G_{n-2} = \frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{5}} + \frac{\alpha^{n-2} - \beta^{n-2}}{\sqrt{5}}$$
$$= \frac{\alpha^{n-2}(\alpha+1) - \beta^{n-2}(\beta+1)}{\sqrt{5}} = \frac{\alpha^n - \beta^n}{\sqrt{5}} = G_n,$$

because  $\alpha^2 = \alpha + 1$ ,  $\beta^2 = \beta + 1$ . Therefore the sequence  $(G_n)_{n \ge 1}$  coincides with the Fibonacci sequence.

Now we establish the estimates.

If 
$$n \ge 1$$
 then  $(-\beta)^n = \frac{1}{\alpha^n} < \alpha^{n-1} = -\alpha^n\beta = \alpha^n(\alpha - 1) = \alpha^{n+1} - \alpha^n$ , so

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \leqslant \frac{\alpha^n + (-\beta)^n}{\sqrt{5}} < \frac{\alpha^{n+1}}{\sqrt{5}}.$$

Similarly, if 
$$n \ge 2$$
 then  $(-\beta)^n = \frac{1}{\alpha^n} < \alpha^{n-2} = -\alpha^{n-1}\beta = \alpha^{n-1}(\alpha - 1)$   
 $= \alpha^n - \alpha^{n-1}$  so  $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \ge \frac{\alpha^n - (-\beta)^n}{\sqrt{5}} > \frac{\alpha^{n-1}}{\sqrt{5}}$ ; this is also true when  $n = 1$ .

For every  $m \ge 1$  let  $I_m = \sum_{n=1}^{\infty} \frac{1}{F_n^{1/m}}$ .

We have:

Lemma 2. For every  $m\geqslant 1,$   $I_m<\infty,$   $I_1< I_2< I_3,...,$  and  $\lim_{m\to\infty}I_m=\infty$  .

*Proof.* We have

$$I_m < \sum_{n=1}^{\infty} \left(\frac{\sqrt{5}}{\alpha^{n-1}}\right)^{1/m} = (\sqrt{5})^{1/m} \sum_{n=1}^{\infty} \left(\frac{1}{\alpha^{1/m}}\right)^{n-1} = \frac{(\sqrt{5})^{1/m} \alpha^{1/m}}{\alpha^{1/m} - 1},$$

noting that  $\frac{1}{\alpha^{1/m}} < 1$ .

Next, we have

$$I_{m-1} = \sum_{n=1}^{\infty} \frac{1}{F_n^{1/m-1}} < \sum_{n=1}^{\infty} \frac{1}{F_n^{1/m}} = I_m.$$

Finally

$$I_{m} = \sum_{n=1}^{\infty} \frac{1}{F_{n}^{1/m}} > \sum_{n=1}^{\infty} \left(\frac{\sqrt{5}}{\alpha^{n+1}}\right)^{1/m} = \frac{(\sqrt{5})^{1/m}}{\alpha^{1/m}} \times \frac{1}{\alpha^{1/m} - 1};$$

thus  $\lim I_m = \infty$ .

PROPOSITION 3. For every positive real number x there exists a unique  $m \ge 1$  such that  $x = \sum_{j=1}^{\infty} \frac{1}{F_{i_j}^{1/m}}$ , but x is not of the form  $\sum_{j=1}^{\infty} \frac{1}{F_{i_j}^{1/(m-1)}}$ .

*Proof.* First we note that each of the sequences  $\left(\frac{1}{F_n^{1/m}}\right)_{n\geq 1}$  is decreasing with limit equal to zero. By the above proposition, there exists  $m\geq 1$  such that  $I_{m-1}< x\leq I_m$  (with  $I_0=0$ ).

We observe that  $\frac{1}{F_n} \leqslant \frac{2}{F_{n+1}} \leqslant \frac{2^m}{F_{n+1}}$  for  $m \geqslant 1$ , because  $F_{n+1} = F_n + F_{n-1} < 2 F_n$ . By proposition 1 and a previous remark, x is representable as indicated, while the last assertion follows from  $x > I_{m-1} = \sum_{i=1}^{\infty} \frac{1}{F_i^{1/m-1}}$ .

The number  $I_1 = \sum_{n=1}^{\infty} \frac{1}{F_n}$  appears to be quite mysterious. As we have seen  $\sqrt{5} < I_1 < \sqrt{5} \frac{\alpha}{\alpha - 1}$ .

4. In 1899, Landau gave an expression of  $I_1$  in terms of Lambert series and Jacobi theta series. The Lambert series is  $L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n}$ ; it is convergent for 0 < x < 1, as easily verified by the ratio test.

Jacobi theta series, which are of crucial importance, for example in the theory of elliptic functions, are defined as follows, for 0 < |q| < 1 and  $z \in C$ :

$$\theta_{1}(z, q) = i \sum_{n=-\infty}^{\infty} (-1)^{n} q^{\left(n-\frac{1}{2}\right)^{2}} e^{(2n-1)\pi i z}$$

$$= 2 q^{1/4} \sin \pi z - 2 q^{9/4} \sin 3\pi z + 2 q^{25/4} \sin 5\pi z - \dots$$

$$\theta_2(z, q) = \sum_{n = -\infty}^{\infty} q^{\left(n + \frac{1}{2}\right)^2} e^{(2n - 1)\pi i z}$$

$$= 2 q^{1/4} \cos \pi i z + 2 q^{9/4} \cos 3\pi z + 2 q^{25/4} \cos 5\pi z + \dots$$

$$\theta_{3}(z, q) = \sum_{n = -\infty}^{\infty} q^{n^{2}} e^{2n\pi i z}$$

$$= 1 + 2q \cos 2\pi z + 2q^{4} \cos 4\pi z + 2q^{9} \cos 6\pi z + \dots$$

$$\theta_4(z, q) = \sum_{n = -\infty}^{\infty} (-1)^n q^{n^2} e^{2n\pi i z}$$

$$= 1 - 2q \cos 2\pi z + 2q^4 \cos 4\pi z - 2q^9 \cos 6\pi z + \dots$$

In particular, we have

$$\theta_1(0, q) = 0$$

$$\theta_2(0, q) = 2q^{1/4} + 2q^{9/4} + 2q^{25/4} + \dots$$

$$\theta_3(0, q) = 1 + 2q + 2q^4 + 2q^9 + \dots$$

$$\theta_4(0, q) = 1 - 2q + 2q^4 - 2q^9 + \dots$$

Now we prove Landau's result:

Proposition 4. We have:

1) 
$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}} = \sqrt{5} \left[ L\left(\frac{3-\sqrt{5}}{2}\right) - L\left(\frac{7-3\sqrt{5}}{2}\right) \right].$$

2) 
$$\sum_{n=0}^{\infty} \frac{1}{F_{2n-1}} = -\sqrt{5} \left( 1 + 2\beta^4 + 2\beta^{16} + 2\beta^{36} + \dots \right) (\beta + \beta^9 + \beta^{25} + \dots)$$
$$= -\frac{\sqrt{5}}{2} \left[ \theta_3(0, \beta) - \theta_2(0, \beta^4) \right] \theta_2(0, \beta^4).$$

**Proof**:

1) We have 
$$\frac{1}{F_n} = \frac{\sqrt{5}}{\alpha^n - \beta^n} = \frac{\sqrt{5}}{\frac{(-1)^n}{\beta^n} - \beta^n} = \frac{\sqrt{5} \beta^n}{(-1)^n - \beta^{2n}}$$

SO

$$\frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{1}{F_{2n}} = \sum_{n=1}^{\infty} \frac{\beta^{2n}}{1 - \beta^{4n}} = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \beta^{(4k+2)n} = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \beta^{(4k+2)n}$$
$$= \sum_{k=0}^{\infty} \frac{\beta^{4k+2}}{1 - \beta^{4k+2}} = \frac{\beta^2}{1 - \beta^2} + \frac{\beta^6}{1 - \beta^6} + \frac{\beta^{10}}{1 - \beta^{10}} + \dots$$

Since  $|\beta| < 1$ , then

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}} = \sqrt{5} \left[ L(\beta^2) - L(\beta^4) \right] = \sqrt{5} \left[ L\left(\frac{3-\sqrt{5}}{2}\right) - L\left(\frac{7-3\sqrt{5}}{2}\right) \right]$$

2) Now we have

$$\frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{1}{F_{2n-1}} = -\sum_{n=0}^{\infty} \frac{\beta^{2n+1}}{1 + \beta^{4n+2}} = -\sum_{n=0}^{\infty} \beta^{2n+1} (1 - \beta^{4n+2} + \beta^{8n+4} + \dots)$$

$$= (-\beta + \beta^3 - \beta^5 + \beta^7 - \beta^9 + \dots)$$

$$+ (-\beta^3 + \beta^9 - \beta^{15} + \beta^{21} - \dots)$$

$$+ (-\beta^5 + \beta^{15} - \beta^{25} + \beta^{35} - \dots)$$

$$+ (-\beta^7 + \beta^{21} - \beta^{35} + \beta^{49} - \dots) + \dots$$

Now we need to determine the coefficient of  $\beta^m$  (for m odd) remarking that since the series is absolutely convergent, its terms may be rearranged. If m is odd and d divides m, then  $\beta^m$  appears in the horizontal line beginning with  $-\beta^{\frac{m}{d}}$  with the sign

$$\begin{cases} + & \text{when } d \equiv 3 \pmod{4} \\ - & \text{when } d \equiv 1 \pmod{4} \end{cases}$$

Thus the coefficient  $\varepsilon_m$  of  $\beta^m$  is  $\varepsilon_m = \delta_3(m) - \delta_1(m)$  where

$$\delta_1(m) = \# \{d \mid 1 \le d \le m, \quad d \mid m \text{ and } d \equiv 1 \pmod{4}\}$$
  
 $\delta_3(m) = \# \{d \mid 1 \le d \le m, \quad d \mid m \text{ and } d \equiv 3 \pmod{4}\}$ 

A well-known result of Jacobi (see Hardy & Wright's book, page 241) relates the difference  $\delta_1(m) - \delta_3(m)$  with the number  $r(m) = r_2(m)$  of representations of m as sums of two squares. Precisely, let r(m) denote the number of pairs (s, t) of integers (including the zero and negative integers) such that  $m = s^2 + t^2$ . Jacobi showed that

$$r(m) = 4 \left[\delta_1(m) - \delta_3(m)\right].$$

It follows that the number r'(m) of pairs (s, t) of integers with  $s > t \ge 0$  and  $m = s^2 + t^2$  is

$$r'(m) = \begin{cases} \frac{r(m)}{8} & \text{when } m \text{ is not a square} \\ \frac{r(m) - 4}{8} + 1 = \frac{r(m) + 4}{8} & \text{when } m \text{ is a square} \end{cases}$$

(the first summand above corresponds to the representation of m as a sum of two non-zero squares).

Therefore

$$\varepsilon_m = -\frac{r(m)}{4} = \begin{cases}
-2r'(m) & \text{when } m \text{ is not a square} \\
-(2r'(m)-1) & \text{when } m \text{ is a square}.
\end{cases}$$

Since m is odd then  $s \not\equiv t \pmod{2}$  and therefore

$$\frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} = \sum_{\substack{m=1\\ m \text{ odd}}}^{\infty} \varepsilon_m \beta^m 
= -2(1+\beta^4+\beta^{16}+\beta^{36}+\ldots)(\beta+\beta^9+\beta^{25}+\ldots) 
+ (\beta+\beta^9+\beta^{25}+\ldots) 
= -(1+2\beta^4+2\beta^{16}+2\beta^{36}+\ldots)(\beta+\beta^9+\beta^{25}+\ldots).$$

So

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} = -\sqrt{5}(1+2\beta^4+2\beta^{16}+2\beta^{36}+\ldots)(\beta+\beta^9+\beta^{25}+\ldots).$$

We may now express this formula in terms of Jacobi series. Namely

$$1 + 2\beta^4 + 2\beta^{16} + 2\beta^{36} + \dots = (1 + 2\beta + 2\beta^4 + 2\beta^9 + 2\beta^{16} + \dots)$$
$$- (2\beta + 2\beta^9 + 2\beta^{25} + \dots) = \theta_3(0, \beta) - \theta_2(0, \beta^4)$$

SO

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} = -\frac{\sqrt{5}}{2} \left[ \theta_3(0, \beta) - \theta_2(0, \beta^4) \right] \theta_2(0, \beta^4). \quad \Box$$

An unpublished formula of Gert Almqvist (1983) gives another expression of  $I_1$  only in terms of Jacobi theta series:

$$I_1 = \frac{\sqrt{5}}{4} \left\{ \left[ \theta_2 \left( 0, -\frac{1}{\beta^2} \right) \right]^2 + \frac{1}{\pi} \int_0^1 \left( \frac{d}{dx} \log \theta_4 \left( x, -\frac{1}{\beta^2} \right) \right) \cot \pi x dx \right\}.$$

Carlitz considered also in 1971 the following numbers:

$$S_k = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} \dots F_{n+k}}$$

so 
$$S_0 = \sum_{n=1}^{\infty} \frac{1}{F_n} = I_1$$
.

Clearly, all the above series are convergent. Carlitz showed that  $S_3, S_7, S_{11}, \ldots \in \mathbb{Q}(\sqrt{5})$ , while  $S_{4k} = r_k + r_k' S_0$  for  $k \ge 1$  and  $r_k, r_k' \in \mathbb{Q}$ . One may ask: what kind of number is  $S_0$ ? Are the numbers  $S_0, S_1, S_2$  algebraically independent?

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