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REPRESENTATION OF REAL NUMBERS BY MEANS OF FIBONACCI NUMBERS

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Dedicated to Professor D.H. Lehmer at the occasion of his eightieth birthday.

The purpose of this note is to derive a new representation of positive real numbers as sums of series involving Fibonacci numbers. This will be an easy application of an old result of Kakeya [4]. The paper concludes with a result of Landau [5], relating the sum $\sum_{n=1}^{\infty} \frac{1}{F_n}$ with values of theta series; we believe it worthwhile to unearth Landau's result, which is now rather inaccessible.

1. Let $(s_i)_{i \geqslant 1}$ be a sequence of positive real numbers, such that $s_1 > s_2 > s_3 > \ldots$ and $\lim_{i \to \infty} s_i = 0$. Let $S = \sum_{i=1}^{\infty} s_i \leqslant \infty$.

We say that $x > 0$ is representable by the sequence $(s_i)_{i \geqslant 1}$ if $x = \sum_{j=1}^{\infty} s_{i_j}$ (with $i_1 < i_2 < i_3 < \ldots$). Then necessarily $x \leqslant S$.

The first result is due to Kakeya; for the sake of completeness, we give a proof:

PROPOSITION 1. *The following conditions are equivalent:*

1) *Every* $x, 0 < x \leqslant S$, *is representable by the sequence* $(s_i)_{i \geqslant 1}$, $x = \sum_{j=1}^{\infty} s_{i_j}$, *where* i_1 *is the smallest index such that* $s_{i_1} < x$.

2) *Every* $x, 0 < x < S$ *is representable by the sequence* $(s_i)_{i \geqslant 1}$.

3) *For every* $n \geqslant 1, s_n \leqslant \sum_{i=n+1}^{\infty} s_i$.

Proof. $1 \to 2$. This is trivial.

$2 \to 3$. If there exists $n \geqslant 1$ such that $s_n > \sum_{i=n+1}^{\infty} s_i$, let x be such that $s_n > x > \sum_{i=n+1}^{\infty} s_i$. By hypothesis, $x = \sum_{j=1}^{\infty} s_{i_j}$ with $i_1 < i_2 < \ldots$ Since $s_n > x > s_{i_1}$, then $n < i_1$, hence $x = \sum_{j=1}^{\infty} s_{i_j} \leqslant \sum_{k=n+1}^{\infty} s_k$, which is absurd.

$3 \to 1$. Since $\lim_{i\to\infty} s_i = 0$, there exists the smallest index i_1 such that $s_{i_1} < x$.

Similarly, there exists the smallest index i_2 such that $i_1 < i_2$ and $s_{i_2} < x - s_{i_1}$.

More generally, for every $n \geqslant 1$ we define i_n to be the smallest index such that $i_{n-1} < i_n$ and $s_{i_n} < x - \sum_{j=1}^{n-1} s_{i_j}$.

Then $x \geqslant \sum_{j=1}^{\infty} s_{i_j}$. Suppose that $x > \sum_{j=1}^{\infty} s_{i_j}$.

We note that there exists N such that if $m \geqslant N$ then $s_{i_m} < x - \sum_{j=1}^{m} s_{i_j}$. Otherwise, there exist infinitely many indices $n_1 < n_2 < n_3 < \dots$ such that $s_{i_{n_k}} \geqslant x - \sum_{j=1}^{n_k} s_{i_j}$. At the limit, we have $0 = \lim_{k\to\infty} s_{i_{n_k}} \geqslant x - \sum_{j=1}^{\infty} s_{i_j} > 0$, and this is a contradiction.

We choose N minimal with above property.

We show: for every $m \geqslant N$, $i_m + 1 = i_{m+1}$. In fact

$$s_{i_m+1} < s_{i_m} < x - \sum_{j=1}^{m} s_{i_j},$$

so by definition of the sequence of indices, $i_m + 1 = i_{m+1}$. Therefore the following sets coincide: $\{i_N, i_N+1, i_N+2, \dots\} = \{i_N, i_{N+1}, i_{N+2}, \dots\}$.

Next we show that $i_N = 1$. If $i_N > 1$ we consider the index $i_N - 1$, and by hypothesis (3):

$$s_{i_N-1} \leqslant \sum_{k=i_N}^{\infty} s_k = \sum_{j=N}^{\infty} s_{i_j} < x - \sum_{j=1}^{N-1} s_{i_j}.$$

We have $i_{N-1} \leqslant i_N - 1 < i_N$. If $i_{N-1} < i_N - 1$ this is impossible, because i_N was defined to be the smallest index such that $i_{N-1} < i_N$ and $s_{i_N} < x - \sum_{j=1}^{N-1} s_{i_j}$. Thus $i_{N-1} = i_N - 1$, that is $s_{i_{N-1}} < x - \sum_{j=1}^{N-1} s_{i_j}$ and this is against the choice of N as minimal with the property indicated.

Thus $i_N = 1$ and $x > \sum_{j=1}^{\infty} s_{i_j} = \sum_{i=1}^{\infty} s_i = S$, against the hypothesis. $\square$

We remark now that if the above conditions are satisfied for the sequence $(s_i)_{i\geqslant 1}$, if $m \geqslant 0$ then every x, $0 < x < S' = \sum_{i=m+1}^{\infty} s_i$ is representable by the sequence $(s_i)_{i\geqslant m+1}$ with i_1 the smallest index such that $m + 1 \leqslant i_1$ and $s_{i_1} < x$.

Indeed, condition (3) holds for $(s_i)_{i \geqslant 1}$ hence also for $(s_i)_{i \geqslant m+1}$. Since $0 < x < S'$, the remark follows from the proposition.

Proposition 1 has been generalized (see for example Fridy [3]). Now we consider the question of unique representation (this was generalized by Brown in [1]).

PROPOSITION 2. *With above notations, the following conditions are equivalent:*

2′) *Every* $x, 0 < x < S$, *has a unique representation* $x = \sum_{j=1}^{\infty} s_{i_j}$.

3′) *For every* $n \geqslant 1, s_n = \sum_{i=n+1}^{\infty} s_i$.

4′) *For every* $n \geqslant 1, s_n = \frac{1}{2^{n-1}} s_1$ *(hence* $S = 2s_1$*).*

Proof. $2' \to 3'$. Suppose there exists $n \geqslant 1$ such that $s_n \neq \sum_{i=n+1}^{\infty} s_i$. Since (2′) implies (2) hence also (3) then $s_n < \sum_{i=n+1}^{\infty} s_i$. Let x be such that $s_n < x < \sum_{i=n+1}^{\infty} s_i$. By the above remark, x is representable by the sequence $\{s_i\}_{i \geqslant n+1}$, that is $x = \sum_{\substack{j=1 \\ k_j \geqslant n+1}}^{\infty} s_{k_j}$. On the other hand, (2′) implies (2), hence also (1) and x has a representation $x = \sum_{j=1}^{\infty} s_{i_j}$, where i_1 is the smallest index such that $s_{i_1} < x$. From $s_n < x$ it follows that $i_1 \leqslant n$ and so x would have two distinct representations, against the hypothesis.

$3' \to 4'$. We have $s_n = s_{n+1} + \sum_{i=n+2}^{\infty} s_i = 2s_{n+1}$ for every $n \geqslant 1$ hence $s_n = \frac{1}{2^{n-1}} s_1$ for every $n \geqslant 1$.

$4' \to 2'$. Suppose that there exists $x, 0 < x < S$ having two distinct representations

$$x = \sum_{j=1}^{\infty} s_{i_j} = \sum_{j=1}^{\infty} s_{k_j}.$$

Let j_0 be the smallest index such that $i_{j_0} \neq k_{j_0}$, say $i_{j_0} < k_{j_0}$. Then $\sum_{j=j_0}^{\infty} s_{i_j} = \sum_{j=j_0}^{\infty} s_{k_j} \leqslant \sum_{n=i_{j_0}+1}^{\infty} s_n$.

By hypothesis, after dividing by s_1, we have

$$\sum_{n=i_{j_0}}^{\infty} \frac{1}{2^n} \geqslant \sum_{j=j_0}^{\infty} 2^{1-k_j} = \sum_{j=j_0}^{\infty} 2^{1-i_j}$$

$$= 2^{1-i_{j_0}} + \sum_{j=j_0+1}^{\infty} 2^{1-i_j} = \sum_{n=i_{j_0}}^{\infty} 2^{-n} + \sum_{j=j_0+1}^{\infty} 2^{1-i_j}$$

hence $\sum_{j=j_0+1}^{\infty} 2^{1-i_j} \leqslant 0$, which is impossible. □

For practical applications, we note:
If $s_n \leqslant 2s_{n+1}$ for every $n \geqslant 1$ then condition (3) is satisfied.
Indeed

$$\sum_{i=n+1}^{\infty} s_i \leqslant 2 \sum_{i=n+1}^{\infty} s_{i+1} = 2 \sum_{i=n+2}^{\infty} s_i, \quad \text{hence} \quad s_{n+1} \leqslant \sum_{i=n+2}^{\infty} s_i$$

and $s_n \leqslant 2\, s_{n+1} \leqslant \sum_{i=n+1}^{\infty} s_i$.

2. Now we give various ways of representing real numbers.

First, the dyadic representation, which may of course be easily obtained directly:

COROLLARY 1. *Every real number* $x, 0 < x < 1$, *may be written uniquely in the form* $x = \sum_{j=1}^{\infty} \frac{1}{2^{n_j}}$ *(with* $1 \leqslant n_1 < n_2 < n_3 < \ldots$*).*

Proof. This has been shown in proposition 2, taking $s_1 = \frac{1}{2}$. □

COROLLARY 2. *Every positive real number* x *may be written in the form* $x = \sum_{j=1}^{\infty} \frac{1}{n_j}$ *(with* $n_1 < n_2 < n_3 < \ldots$*).*

Proof. We consider the sequence $\left(\frac{1}{n}\right)_{n \geqslant 1}$, which is decreasing with limit equal to zero, and we note that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ and $\frac{1}{n} \leqslant \frac{2}{n+1}$ for every $n \geqslant 1$.

Thus by Kakeya's theorem and the above remark, every $x > 0$ is representable as indicated. □

COROLLARY 3. *Every positive real number* x *may be written as* $x = \sum_{j=1}^{\infty} \frac{1}{p_{i_j}}$ *(where* $p_1 < p_2 < p_3 < \ldots$ *is the increasing sequence of prime numbers).*

Proof. We consider the sequence $\left(\frac{1}{p_i}\right)_{i \geqslant 1}$, which is decreasing with limit equal to zero. As Euler proved $\sum_{i=1}^{\infty} \frac{1}{p_i} = \infty$. By Tschebycheff's theorem (proof of Bertrand's "postulate") there is a prime in each interval $(n, 2n)$; thus $p_{i+1} < 2\, p_i$ and $\frac{1}{p_i} < \frac{2}{p_{i+1}}$ for every $i \geqslant 1$. By Kakeya's theorem and the above remark, every $x > 0$ is representable as indicated. □

3. Now we shall represent real numbers by means of Fibonacci numbers and we begin giving some properties of these numbers.

The Fibonacci numbers are: $F_1 = F_2 = 1$ and for every $n \geqslant 3$, F_n is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$.

Thus the sequence of Fibonacci numbers is

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

In the following proposition, we give a closed form expression for the Fibonacci numbers; this is due to Binet (1843).

Let $\alpha = \frac{\sqrt{5}+1}{2}$ (the golden number) and $\beta = -\frac{\sqrt{5}-1}{2}$, so $\alpha + \beta = 1$, $\alpha\beta = -1$, thus α, β are the roots of $X^2 - X - 1 = 0$ and $-1 < \beta < 0 < 1 < \alpha$.

We have

LEMMA 1. *For every* $n \geqslant 1$, $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ *and* $\frac{\alpha^{n-1}}{\sqrt{5}} < F_n < \frac{\alpha^{n+1}}{\sqrt{5}}$.

Proof. We consider the sequence of numbers $G_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ for $n \geqslant 1$.

Then $G_1 = G_2 = 1$; moreover

$$G_{n-1} + G_{n-2} = \frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{5}} + \frac{\alpha^{n-2} - \beta^{n-2}}{\sqrt{5}}$$

$$= \frac{\alpha^{n-2}(\alpha+1) - \beta^{n-2}(\beta+1)}{\sqrt{5}} = \frac{\alpha^n - \beta^n}{\sqrt{5}} = G_n,$$

because $\alpha^2 = \alpha + 1$, $\beta^2 = \beta + 1$. Therefore the sequence $(G_n)_{n\geqslant 1}$ coincides with the Fibonacci sequence.

Now we establish the estimates.

If $n \geqslant 1$ then $(-\beta)^n = \dfrac{1}{\alpha^n} < \alpha^{n-1} = -\alpha^n\beta = \alpha^n(\alpha-1) = \alpha^{n+1} - \alpha^n$, so

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \leqslant \frac{\alpha^n + (-\beta)^n}{\sqrt{5}} < \frac{\alpha^{n+1}}{\sqrt{5}}.$$

Similarly, if $n \geqslant 2$ then $(-\beta)^n = \dfrac{1}{\alpha^n} < \alpha^{n-2} = -\alpha^{n-1}\beta = \alpha^{n-1}(\alpha-1)$ $= \alpha^n - \alpha^{n-1}$ so $F_n = \dfrac{\alpha^n - \beta^n}{\sqrt{5}} \geqslant \dfrac{\alpha^n - (-\beta)^n}{\sqrt{5}} > \dfrac{\alpha^{n-1}}{\sqrt{5}}$; this is also true when $n = 1$. □

For every $m \geqslant 1$ let $I_m = \sum_{n=1}^{\infty} \dfrac{1}{F_n^{1/m}}$.

We have:

Lemma 2. *For every $m \geqslant 1$, $I_m < \infty$, $I_1 < I_2 < I_3, \ldots,$ and $\lim_{m\to\infty} I_m = \infty$.*

Proof. We have

$$I_m < \sum_{n=1}^{\infty} \left(\frac{\sqrt{5}}{\alpha^{n-1}}\right)^{1/m} = (\sqrt{5})^{1/m} \sum_{n=1}^{\infty} \left(\frac{1}{\alpha^{1/m}}\right)^{n-1} = \frac{(\sqrt{5})^{1/m}\,\alpha^{1/m}}{\alpha^{1/m} - 1},$$

noting that $\dfrac{1}{\alpha^{1/m}} < 1$.

Next, we have

$$I_{m-1} = \sum_{n=1}^{\infty} \frac{1}{F_n^{1/m-1}} < \sum_{n=1}^{\infty} \frac{1}{F_n^{1/m}} = I_m.$$

Finally

$$I_m = \sum_{n=1}^{\infty} \frac{1}{F_n^{1/m}} > \sum_{n=1}^{\infty} \left(\frac{\sqrt{5}}{\alpha^{n+1}}\right)^{1/m} = \frac{(\sqrt{5})^{1/m}}{\alpha^{1/m}} \times \frac{1}{\alpha^{1/m} - 1};$$

thus $\lim_{m\to\infty} I_m = \infty$. □

PROPOSITION 3. *For every positive real number* x *there exists a unique* $m \geqslant 1$ *such that* $x = \sum_{j=1}^{\infty} \frac{1}{F_{i_j}^{1/m}}$, *but* x *is not of the form* $\sum_{j=1}^{\infty} \frac{1}{F_{i_j}^{1/(m-1)}}$.

Proof. First we note that each of the sequences $\left(\frac{1}{F_n^{1/m}}\right)_{n \geqslant 1}$ is decreasing with limit equal to zero. By the above proposition, there exists $m \geqslant 1$ such that $I_{m-1} < x \leqslant I_m$ (with $I_0 = 0$).

We observe that $\frac{1}{F_n} \leqslant \frac{2}{F_{n+1}} \leqslant \frac{2^m}{F_{n+1}}$ for $m \geqslant 1$, because $F_{n+1} = F_n + F_{n-1} < 2\,F_n$. By proposition 1 and a previous remark, x is representable as indicated, while the last assertion follows from $x > I_{m-1} = \sum_{i=1}^{\infty} \frac{1}{F_i^{1/m-1}}$.

□

The number $I_1 = \sum_{n=1}^{\infty} \frac{1}{F_n}$ appears to be quite mysterious. As we have seen $\sqrt{5} < I_1 < \sqrt{5}\,\frac{\alpha}{\alpha - 1}$.

4. In 1899, Landau gave an expression of I_1 in terms of Lambert series and Jacobi theta series. The Lambert series is $L(x) = \sum_{n=1}^{\infty} \frac{x^n}{1 - x^n}$; it is convergent for $0 < x < 1$, as easily verified by the ratio test.

Jacobi theta series, which are of crucial importance, for example in the theory of elliptic functions, are defined as follows, for $0 < |q| < 1$ and $z \in C$:

$$\theta_1(z, q) = i \sum_{n=-\infty}^{\infty} (-1)^n q^{\left(n - \frac{1}{2}\right)^2} e^{(2n-1)\pi i z}$$
$$= 2\,q^{1/4} \sin \pi z - 2\,q^{9/4} \sin 3\pi z + 2\,q^{25/4} \sin 5\pi z - \dots$$

$$\theta_2(z, q) = \sum_{n=-\infty}^{\infty} q^{\left(n + \frac{1}{2}\right)^2} e^{(2n-1)\pi i z}$$
$$= 2\,q^{1/4} \cos \pi i z + 2\,q^{9/4} \cos 3\pi z + 2\,q^{25/4} \cos 5\pi z + \dots$$

$$\theta_3(z, q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2n\pi i z}$$
$$= 1 + 2q \cos 2\pi z + 2q^4 \cos 4\pi z + 2q^9 \cos 6\pi z + \dots$$

$$\theta_4(z, q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2n\pi i z}$$
$$= 1 - 2q \cos 2\pi z + 2q^4 \cos 4\pi z - 2q^9 \cos 6\pi z + \dots$$

In particular, we have

$$\theta_1(0, q) = 0$$
$$\theta_2(0, q) = 2q^{1/4} + 2q^{9/4} + 2q^{25/4} + \ldots$$
$$\theta_3(0, q) = 1 + 2q + 2q^4 + 2q^9 + \ldots$$
$$\theta_4(0, q) = 1 - 2q + 2q^4 - 2q^9 + \ldots$$

Now we prove Landau's result:

PROPOSITION 4. *We have:*

1) $$\sum_{n=1}^{\infty} \frac{1}{F_{2n}} = \sqrt{5}\left[L\left(\frac{3-\sqrt{5}}{2}\right) - L\left(\frac{7-3\sqrt{5}}{2}\right)\right].$$

2) $$\sum_{n=0}^{\infty} \frac{1}{F_{2n-1}} = -\sqrt{5}\,(1+2\beta^4+2\beta^{16}+2\beta^{36}+\ldots)(\beta+\beta^9+\beta^{25}+\ldots)$$
$$= -\frac{\sqrt{5}}{2}[\theta_3(0, \beta) - \theta_2(0, \beta^4)]\,\theta_2(0, \beta^4).$$

Proof:

1) We have $\dfrac{1}{F_n} = \dfrac{\sqrt{5}}{\alpha^n - \beta^n} = \dfrac{\sqrt{5}}{\dfrac{(-1)^n}{\beta^n} - \beta^n} = \dfrac{\sqrt{5}\,\beta^n}{(-1)^n - \beta^{2n}}$

so

$$\frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{1}{F_{2n}} = \sum_{n=1}^{\infty} \frac{\beta^{2n}}{1 - \beta^{4n}} = \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \beta^{(4k+2)n} = \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} \beta^{(4k+2)n}$$
$$= \sum_{k=0}^{\infty} \frac{\beta^{4k+2}}{1 - \beta^{4k+2}} = \frac{\beta^2}{1-\beta^2} + \frac{\beta^6}{1-\beta^6} + \frac{\beta^{10}}{1-\beta^{10}} + \ldots$$

Since $|\beta| < 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{F_{2n}} = \sqrt{5}\left[L(\beta^2) - L(\beta^4)\right] = \sqrt{5}\left[L\left(\frac{3-\sqrt{5}}{2}\right) - L\left(\frac{7-3\sqrt{5}}{2}\right)\right].$$

2) Now we have

$$\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\frac{1}{F_{2n-1}} = -\sum_{n=0}^{\infty}\frac{\beta^{2n+1}}{1+\beta^{4n+2}} = -\sum_{n=0}^{\infty}\beta^{2n+1}(1-\beta^{4n+2}+\beta^{8n+4}+\ldots)$$

$$\begin{aligned}&= (-\beta+\beta^3-\beta^5+\beta^7-\beta^9+\ldots)\\ &+ (-\beta^3+\beta^9-\beta^{15}+\beta^{21}-\ldots)\\ &+ (-\beta^5+\beta^{15}-\beta^{25}+\beta^{35}-\ldots)\\ &+ (-\beta^7+\beta^{21}-\beta^{35}+\beta^{49}-\ldots)+\ldots\end{aligned}$$

Now we need to determine the coefficient of β^m (for m odd) remarking that since the series is absolutely convergent, its terms may be rearranged.

If m is odd and d divides m, then β^m appears in the horizontal line beginning with $-\beta^{\frac{m}{d}}$ with the sign

$$\begin{cases} + & \text{when} \quad d \equiv 3 \pmod 4 \\ - & \text{when} \quad d \equiv 1 \pmod 4\,. \end{cases}$$

Thus the coefficient ε_m of β^m is $\varepsilon_m = \delta_3(m) - \delta_1(m)$ where

$$\delta_1(m) = \#\{d \mid 1 \leqslant d \leqslant m, \quad d \mid m \quad \text{and} \quad d \equiv 1 \pmod 4\}$$
$$\delta_3(m) = \#\{d \mid 1 \leqslant d \leqslant m, \quad d \mid m \quad \text{and} \quad d \equiv 3 \pmod 4\}$$

A well-known result of Jacobi (see Hardy & Wright's book, page 241) relates the difference $\delta_1(m) - \delta_3(m)$ with the number $r(m) = r_2(m)$ of representations of m as sums of two squares. Precisely, let $r(m)$ denote the number of pairs (s, t) of integers (including the zero and negative integers) such that $m = s^2 + t^2$. Jacobi showed that

$$r(m) = 4\,[\delta_1(m) - \delta_3(m)]\,.$$

It follows that the number $r'(m)$ of pairs (s, t) of integers with $s > t \geqslant 0$ and $m = s^2 + t^2$ is

$$r'(m) = \begin{cases} \dfrac{r(m)}{8} & \text{when } m \text{ is not a square} \\[2ex] \dfrac{r(m)-4}{8} + 1 = \dfrac{r(m)+4}{8} & \text{when } m \text{ is a square} \end{cases}$$

(the first summand above corresponds to the representation of m as a sum of two non-zero squares).

Therefore

$$\varepsilon_m = -\frac{r(m)}{4} = \begin{cases} -2r'(m) & \text{when } m \text{ is not a square} \\ \\ -(2r'(m)-1) & \text{when } m \text{ is a square}\,. \end{cases}$$

Since m is odd then $s \not\equiv t \pmod 2$ and therefore

$$\begin{aligned} \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} &= \sum_{\substack{m=1 \\ m \text{ odd}}}^{\infty} \varepsilon_m \beta^m \\ &= -2(1+\beta^4+\beta^{16}+\beta^{36}+\ldots)(\beta+\beta^9+\beta^{25}+\ldots) \\ &\quad + (\beta+\beta^9+\beta^{25}+\ldots) \\ &= -(1+2\beta^4+2\beta^{16}+2\beta^{36}+\ldots)(\beta+\beta^9+\beta^{25}+\ldots)\,. \end{aligned}$$

So

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} = -\sqrt{5}(1+2\beta^4+2\beta^{16}+2\beta^{36}+\ldots)(\beta+\beta^9+\beta^{25}+\ldots)\,.$$

We may now express this formula in terms of Jacobi series. Namely

$$\begin{aligned} 1 + 2\beta^4 + 2\beta^{16} + 2\beta^{36} + \ldots &= (1+2\beta+2\beta^4+2\beta^9+2\beta^{16}+\ldots) \\ &\quad - (2\beta+2\beta^9+2\beta^{25}+\ldots) = \theta_3(0, \beta) - \theta_2(0, \beta^4) \end{aligned}$$

so

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} = -\frac{\sqrt{5}}{2}\left[\theta_3(0, \beta) - \theta_2(0, \beta^4)\right]\theta_2(0, \beta^4)\,. \qquad \square$$

An unpublished formula of Gert Almqvist (1983) gives another expression of I_1 only in terms of Jacobi theta series:

$$I_1 = \frac{\sqrt{5}}{4}\left\{\left[\theta_2\left(0, -\frac{1}{\beta^2}\right)\right]^2 + \frac{1}{\pi}\int_0^1 \left(\frac{d}{dx}\log\theta_4\left(x, -\frac{1}{\beta^2}\right)\right)\cot \pi x dx\right\}.$$

Carlitz considered also in 1971 the following numbers:

$$S_k = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} \cdots F_{n+k}}$$

so $S_0 = \sum_{n=1}^{\infty} \frac{1}{F_n} = I_1$.

Clearly, all the above series are convergent. Carlitz showed that $S_3, S_7, S_{11}, \ldots \in \mathbf{Q}(\sqrt{5})$, while $S_{4k} = r_k + r'_k S_0$ for $k \geqslant 1$ and $r_k, r'_k \in \mathbf{Q}$.

One may ask: what kind of number is S_0? Are the numbers S_0, S_1, S_2 algebraically independent?

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