

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 30 (1984)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE ARITHMETIC-GEOMETRIC MEAN OF GAUSS
Autor: Cox, David A.
Kapitel: Introduction
DOI: <https://doi.org/10.5169/seals-53831>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 27.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

THE ARITHMETIC-GEOMETRIC MEAN OF GAUSS

by David A. Cox

INTRODUCTION

The arithmetic-geometric mean of two numbers a and b is defined to be the common limit of the two sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ determined by the algorithm

$$(0.1) \quad \begin{aligned} a_0 &= a, & b_0 &= b, \\ a_{n+1} &= (a_n + b_n)/2, & b_{n+1} &= (a_n b_n)^{1/2}, \quad n = 0, 1, 2, \dots \end{aligned}$$

Note that a_1 and b_1 are the respective arithmetic and geometric means of a and b , a_2 and b_2 the corresponding means of a_1 and b_1 , etc. Thus the limit

$$(0.2) \quad M(a, b) = \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$$

really does deserve to be called the arithmetic-geometric mean of a and b . This algorithm first appeared in a paper of Lagrange, but it was Gauss who really discovered the amazing depth of this subject. Unfortunately, Gauss published little on the agM (his abbreviation for the arithmetic-geometric mean) during his lifetime. It was only with the publication of his collected works [12] between 1868 and 1927 that the full extent of his work became apparent. Immediately after the last volume appeared, several papers (see [15] and [35]) were written to bring this material to a wider mathematical audience. Since then, little has been done, and only the more elementary properties of the agM are widely known today.

In § 1 we review these elementary properties, where a and b are positive real numbers and the square root in (0.1) is also positive. The convergence of the algorithm is easy to see, though less obvious is the connection between the agM and certain elliptic integrals. As an application, we use $M(\sqrt{2}, 1)$ to determine the arc length of the lemniscate. In § 2, we allow a and b to be complex numbers, and the level of difficulty changes dramatically.

The convergence of the algorithm is no longer obvious, and as might be expected, the square root in (0.1) causes trouble. In fact, $M(a, b)$ becomes a multiple valued function, and in order to determine the relation between the various values, we will need to “uniformize” the agM using quotients of the classical Jacobian theta functions, which are modular functions for certain congruence subgroups of level four in $SL(2, \mathbf{Z})$. The amazing fact is that Gauss knew all of this! Hence in § 3 we explore some of the history of these ideas. The topics encountered will range from Bernoulli’s study of elastic rods (the origin of the lemniscate) to Gauss’ famous mathematical diary and his work on secular perturbations (the only article on the agM published in his lifetime).

I would like to thank my colleagues David Armacost and Robert Breusch for providing translations of numerous passages originally in Latin or German. Thanks also go to Don O’Shea for suggesting the wonderfully quick proof of (2.2) given in § 2.

1. THE ARITHMETIC-GEOMETRIC MEAN OF REAL NUMBERS

When a and b are positive real numbers, the properties of the agM $M(a, b)$ are well known (see, for example, [5] and [26]). We will still give complete proofs of these properties so that the reader can fully appreciate the difficulties we encounter in § 2.

We will assume that $a \geq b > 0$, and we let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be as in (0.1), where b_{n+1} is always the positive square root of $a_n b_n$. The usual inequality between the arithmetic and geometric means,

$$(a+b)/2 \geq (ab)^{1/2},$$

immediately implies that $a_n \geq b_n$ for all $n \geq 0$. Actually, much more is true: we have

$$(1.1) \quad a \geq a_1 \geq \dots \geq a_n \geq a_{n+1} \geq \dots \geq b_{n+1} \geq b_n \geq \dots \geq b_1 \geq b$$

$$(1.2) \quad 0 \leq a_n - b_n \leq 2^{-n}(a-b).$$

To prove (1.1), note that $a_n \geq b_n$ and $a_{n+1} \geq b_{n+1}$ imply

$$a_n \geq (a_n + b_n)/2 = a_{n+1} \geq b_{n+1} = (a_n b_n)^{1/2} \geq b_n,$$

and (1.1) follows. From $b_{n+1} \geq b_n$ we obtain

$$a_{n+1} - b_{n+1} \leq a_{n+1} - b_n = 2^{-1}(a_n - b_n),$$