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THE ARITHMETIC-GEOMETRIC MEAN OF GAUSS

by David A. Cox

Introduction

The arithmetic-geometric mean of two numbers a and b is defined to be the common limit of the two sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ determined by the algorithm

$$(0.1) \qquad \begin{aligned} a_0 &= a\,, & b_0 &= b\,, \\ a_{n+1} &= (a_n+b_n)/2\,, & b_{n+1} &= (a_n b_n)^{1/2}\,, \quad n = 0, 1, 2, \ldots\,. \end{aligned}$$

Note that a_1 and b_1 are the respective arithmetic and geometric means of a and b, a_2 and b_2 the corresponding means of a_1 and b_1, etc. Thus the limit

$$(0.2) \qquad M(a, b) = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n$$

really does deserve to be called the arithmetic-geometric mean of a and b. This algorithm first appeared in a paper of Lagrange, but it was Gauss who really discovered the amazing depth of this subject. Unfortunately, Gauss published little on the agM (his abbreviation for the arithmetic-geometric mean) during his lifetime. It was only with the publication of his collected works [12] between 1868 and 1927 that the full extent of his work became apparent. Immediately after the last volume appeared, several papers (see [15] and [35]) were written to bring this material to a wider mathematical audience. Since then, little has been done, and only the more elementary properties of the agM are widely known today.

In § 1 we review these elementary properties, where a and b are positive real numbers and the square root in (0.1) is also positive. The convergence of the algorithm is easy to see, though less obvious is the connection between the agM and certain elliptic integrals. As an application, we use $M(\sqrt{2}, 1)$ to determine the arc length of the lemniscate. In § 2, we allow a and b to be complex numbers, and the level of difficulty changes dramatically.

The convergence of the algorithm is no longer obvious, and as might be expected, the square root in (0.1) causes trouble. In fact, $M(a, b)$ becomes a multiple valued function, and in order to determine the relation between the various values, we will need to "uniformize" the agM using quotients of the classical Jacobian theta functions, which are modular functions for certain congruence subgroups of level four in $SL(2, \mathbf{Z})$. The amazing fact is that Gauss knew all of this! Hence in § 3 we explore some of the history of these ideas. The topics encountered will range from Bernoulli's study of elastic rods (the origin of the lemniscate) to Gauss' famous mathematical diary and his work on secular perturbations (the only article on the agM published in his lifetime).

I would like to thank my colleagues David Armacost and Robert Breusch for providing translations of numerous passages originally in Latin or German. Thanks also go to Don O'Shea for suggesting the wonderfully quick proof of (2.2) given in § 2.

1. The arithmetic-geometric mean of real numbers

When a and b are positive real numbers, the properties of the agM $M(a, b)$ are well known (see, for example, [5] and [26]). We will still give complete proofs of these properties so that the reader can fully appreciate the difficulties we encounter in § 2.

We will assume that $a \geqslant b > 0$, and we let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be as in (0.1), where b_{n+1} is always the positive square root of $a_n b_n$. The usual inequality between the arithmetic and geometric means,

$$(a+b)/2 \geqslant (ab)^{1/2},$$

immediately implies that $a_n \geqslant b_n$ for all $n \geqslant 0$. Actually, much more is true: we have

$$a \geqslant a_1 \geqslant \ldots \geqslant a_n \geqslant a_{n+1} \geqslant \ldots \geqslant b_{n+1} \geqslant b_n \geqslant \ldots \geqslant b_1 \geqslant b \tag{1.1}$$

$$0 \leqslant a_n - b_n \leqslant 2^{-n}(a-b). \tag{1.2}$$

To prove (1.1), note that $a_n \geqslant b_n$ and $a_{n+1} \geqslant b_{n+1}$ imply

$$a_n \geqslant (a_n+b_n)/2 = a_{n+1} \geqslant b_{n+1} = (a_n b_n)^{1/2} \geqslant b_n,$$

and (1.1) follows. From $b_{n+1} \geqslant b_n$ we obtain

$$a_{n+1} - b_{n+1} \leqslant a_{n+1} - b_n = 2^{-1}(a_n - b_n),$$