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and the sets of Theorem 4 (b) are all Borel. Since Borel sets have the Property of Baire, each A_α may be written as $R_\alpha \Delta M_\alpha$ where R_α is open and M_α is meager. But each A_α , being Borel equidecomposable to all of H^2 , is nonmeager, whence each R_α is nonempty. It follows that the R_α are pairwise disjoint, which contradicts the separability of H^2 . A similar argument shows that the sets cannot all be Lebesgue measurable either.

Let us point out how the proof of Theorem 9 breaks down in hyperbolic space. Theorem 9 is based on the fact that \mathbf{R}^n is a union of countably many sets B_r of finite Lebesgue measure satisfying: for any isometry σ , $m(B_r \Delta \sigma(B_r))/m(B_r) \rightarrow 0$ as $r \rightarrow \infty$. Simply let B_r be the ball of radius r centered at the origin. Because Theorem 9 is false for H^n if $n \geq 2$, there can be no such sequence of almost invariant sets of finite (hyperbolic) measure in H^n .

§ 9. LINEAR TRANSFORMATIONS OF THE EUCLIDEAN PLANE

Paradoxical decompositions in the plane are possible if one allows the use of area-preserving affine transformations. This was first realized by von Neumann [31], who showed that a square is paradoxical using this expansion of the isometry group. In fact, it is sufficient to consider the group generated by $SL_2(\mathbf{Z})$ and all translations; see [39]. In this section we discuss how the results of this paper are affected by considering linear, or affine, transformations instead of just isometries.

Let us consider the action of $SL_2(\mathbf{R})$ on $\mathbf{R}^2 \setminus \{0\}$. The two matrices, $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ freely generate a subgroup of $SL_2(\mathbf{Z})$, no nonidentity element of which has a fixed point in $\mathbf{R}^2 \setminus \{0\}$; this follows from the result of Magnus and Neumann mentioned in § 6, since an element of $SL_2(\mathbf{Z})$ has a nonzero fixed point in \mathbf{R}^2 if and only if it has trace 2. It follows by the technique of § 4 that $SL_2(\mathbf{R})$ has a free subgroup with a perfect set of free generators whose action on $\mathbf{R}^2 \setminus \{0\}$ is fixed-point free. Therefore the action of $SL_2(\mathbf{R})$ on $\mathbf{R}^2 \setminus \{0\}$ satisfies all the conclusions of Theorems 4 and 6.

Using techniques of functional analysis, J. Rosenblatt and R. Kallman (unpublished) have recently shown that the Lebesgue measurable subsets of $\mathbf{R}^n \setminus \{0\}$ ($n \geq 2$) do not bear a finitely additive, $SL_n(\mathbf{Z})$ -invariant measure of total measure one. (For $n \geq 3$ this uses the fact that $SL_n(\mathbf{Z})$ has Kazhdan's Property T, while the \mathbf{R}^2 case uses specific facts about representations of

$SL_2(\mathbf{Z})$; see [41; Theorem 11.17].) Thus Theorem 9 does not extend to area-preserving affine transformations. It would be interesting if a paradoxical decomposition of $\mathbf{R}^2 \setminus \{0\}$ using measurable sets, similar to the one illustrated in § 8, could be explicitly constructed. Some sort of paradoxical decomposition using measurable pieces must exist, by a general theorem of Tarski (see [41]), but it is not known if one using just four pieces exists. On the other hand, Belley and Prasad [4] have shown that there is a finitely additive measure on a certain (not too small) Boolean algebra of Borel subsets of \mathbf{R}^n that has total measure one and is invariant under all nonsingular affine transformations of \mathbf{R}^n (not just the measure-preserving ones).

Finally, we mention some unsolved problems about the existence of nice free groups of affine, area-preserving transformations, positive solutions to which would yield (via Theorems 4-6) paradoxical decompositions of \mathbf{R}^n . Let $A_n(\mathbf{R})$ denote the group of affine transformations of \mathbf{R}^n , i.e., transformations of the form TL , where T is a translation and $L \in GL_n(\mathbf{R})$. Let $SA_n(\mathbf{R})$ be the subgroup obtained by restricting L to $SL_n(\mathbf{R})$, and let $SA_n(\mathbf{Z})$ consist of those TL where $L \in SL_n(\mathbf{Z})$ and T is a translation by a vector in \mathbf{Z}^n . Note that $SA_n(\mathbf{Z})$ acts on \mathbf{Z}^n . Since $G(\mathbf{R}^3) \subseteq SA_3(\mathbf{R})$, Theorem 1 yields that $SA_3(\mathbf{R})$ has a free non-Abelian subgroup whose action on \mathbf{R}^3 is fixed-point free. Consideration of \mathbf{Z}^3 instead of \mathbf{R}^3 leads to problem 1 below. Problem 2 is an attempt to get a version of these results for \mathbf{R}^2 (rather than $\mathbf{R}^2 \setminus \{0\}$, which is treated at the beginning of this section). Only local commutativity is sought because of part (b) of the proposition below. Since $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and its transpose freely generate a group of rank two, so do the two transformations:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence perhaps the subgroup of $SA_2(\mathbf{Z})$ which these two transformations generate solves Problem 2 affirmatively. But we are unable to show that this subgroup is locally commutative.

Problems.

1. Does $SA_3(\mathbf{Z})$ have a free subgroup of rank two which is fixed-point free on \mathbf{Z}^3 ?

2. Does $SA_2(\mathbf{R})$ (or $SA_2(\mathbf{Z})$) have a subgroup of rank two which is locally commutative in its action on \mathbf{R}^2 (or on \mathbf{Z}^2)?

PROPOSITION 10.

(a) If $TL \in A_n(\mathbf{R})$ and TL has no fixed points in \mathbf{R}^n , then L has $+1$ as an eigenvalue, i.e., L has a fixed point in $\mathbf{R}^n \setminus \{0\}$.

(b) If G is a subgroup of $SA_2(\mathbf{R})$ which is fixed-point free on \mathbf{R}^2 then G is solvable.

Proof.

(a) Suppose T is a translation by the vector v . Since $L(x) + v = x$ has no solution, the same is true of $(L - I)(x) = -v$, and therefore $\det(L - I) = 0$, i.e., 1 is an eigenvalue of L .

(b) Let $\sigma = TL$ and $\tau = T'L'$ be in G . Then $\sigma\tau = T''LL'$ so part (a) yields that each of L , L' , LL' has 1 as an eigenvalue. Since these are 2×2 matrices with determinant 1 , this implies that all have trace 2 . Hence, choosing an appropriate basis, we have $L = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and $L' = \begin{pmatrix} \alpha & \beta \\ \gamma & 2 - \alpha \end{pmatrix}$.

Then $LL' = \begin{pmatrix} \alpha + b\gamma & * \\ * & 2 - \alpha \end{pmatrix}$, and the trace of the latter being 2 yields that $b\gamma = 0$. But if either b or γ equal zero, then L and L' commute, which implies that the commutators $\sigma\tau\sigma^{-1}\tau^{-1}$ and $\sigma^{-1}\tau^{-1}\sigma\tau$ are pure translations. Hence $[[G, G], [G, G]]$ is the identity subgroup, i.e., G is solvable.

Part (b) of the Proposition shows why there is no fixed-point free, non-Abelian free subgroup of $SA_2(\mathbf{R})$. But the following problem is unsolved.

Problem 3. Does there exist a free non-Abelian semigroup in $SA_2(\mathbf{R})$ (or $SA_2(\mathbf{Z})$) whose action on \mathbf{R}^2 is fixed-point free?

Part (a) of Proposition 10 brings to light a distinction between the groups $G(\mathbf{R}^n)$ according as n is even or odd. The proof of Theorem 1 for \mathbf{R}^3 (§ 5) is essentially the same as the proof for S^{2n+1} given in § 4. Precisely, it is shown that $A = \{\sigma \in G(\mathbf{R}^3) : \sigma \text{ has a fixed point in } \mathbf{R}^3\}$ is nowhere dense and, in fact, each $R_w = f_w^{-1}(A)$ is nowhere dense in the appropriate product, where w is any group word in finitely many variables. While this is sufficient to get the existence of perfect free generating sets of fixed-point free subgroups in \mathbf{R}^3 and beyond, the set A can fail to be nowhere dense in the higher dimensions. Indeed, consider \mathbf{R}^{2n} , $n \geq 1$. Letting $\pi: G(\mathbf{R}^{2n}) \rightarrow SO_{2n}$ be the canonical homomorphism, it follows from part (a) of Proposition 10 that $G(\mathbf{R}^{2n}) \setminus A \subseteq \pi^{-1}(B)$, where $B = \{L \in SO_{2n} : L \text{ has } 1 \text{ as an eigenvalue}\}$. It is easy to see that B is nowhere dense and it follows that the same is true of $\pi^{-1}(B)$; i.e., A has a nowhere dense complement. In odd

dimensions, however, the situation in \mathbf{R}^3 is typical, as the following proposition shows.

PROPOSITION 11. *If $n \geq 1$ is odd then $A = \{\sigma \in G(\mathbf{R}^n) : \sigma \text{ has a fixed point in } \mathbf{R}^n\}$ is a nowhere dense subset of $G(\mathbf{R}^n)$.*

Proof. It is an easy linear algebra exercise (generalizing Proposition 10 (a) above) to see that $\sigma = TL$ has a fixed point in \mathbf{R}^n if and only if the translation vector of T is orthogonal to all vectors fixed by L . Since there is a basis for the fixed space of L that consists of vectors whose entries are polynomials in the entries of L (Gaussian elimination and scaling), this latter condition on TL is equivalent to the vanishing of a polynomial in the entries of σ . But the condition is not universally true in $G(\mathbf{R}^n)$ since any pure translation has no fixed points; therefore the technique introduced in § 4 implies that A is nowhere dense, as desired.

This proposition, in exactly the same cases, is valid for SO_{n+1} 's action on S^n (see § 4). The following extension of these results is a refinement of the theorems on the existence of free, fixed-point free groups of isometries of rank m : it shows that in these cases almost all (from the category point of view) m -tuples of isometries are free generators of fixed-point free groups of isometries.

PROPOSITION 12. *Suppose n is odd and $n \geq 3$, and X is one of \mathbf{R}^n or S^n . Then any m elements of $G(X)$, with the exception of a meager set in $G(X)^m$, are free generators of a fixed-point free subgroup of $G(X)$.*

Proof. For the spherical case this follows from § 4, where it was shown that $\cup\{R_w : w \text{ is a group word in } m \text{ variables}\}$ is comeager. The Euclidean case is proved by observing (see Proposition 11's proof and § 5) that there is a function p that is a polynomial in the entries of $\sigma_1, \dots, \sigma_m$ such that $p = 0$ if and only if $f_w(\sigma_1, \dots, \sigma_m) \in A$. Since, by the rank two case of Theorem 1 (a), f is not identically zero, $f_w^{-1}(A)$ is nowhere dense. Therefore the union over all words in m variables is meager, as desired.