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and the sets of Theorem 4 (b) are all Borel. Since Borel sets have the Property of Baire, each  $A_{\alpha}$  may be written as  $R_{\alpha} \Delta M_{\alpha}$  where  $R_{\alpha}$  is open and  $M_{\alpha}$  is meager. But each  $A_{\alpha}$ , being Borel equidecomposable to all of  $H^2$ , is nonmeager, whence each  $R_{\alpha}$  is nonempty. It follows that the  $R_{\alpha}$  are pairwise disjoint, which contradicts the separability of  $H^2$ . A similar argument shows that the sets cannot all be Lebesgue measurable either.

Let us point out how the proof of Theorem 9 breaks down in hyperbolic space. Theorem 9 is based on the fact that  $\mathbb{R}^n$  is a union of countably many sets  $B_r$ , of finite Lebesgue measure satisfying: for any isometry  $\sigma$ ,  $m(B_r\Delta\sigma(B_r))/m(B_r) \to 0$  as  $r \to \infty$ . Simply let  $B_r$  be the ball of radius r centered at the origin. Because Theorem 9 is false for  $H^n$  if  $n \ge 2$ , there can be no such sequence of almost invariant sets of finite (hyperbolic) measure in  $H^n$ .

## § 9. Linear Transformations of the Euclidean Plane

Paradoxical decompositions in the plane are possible if one allows the use of area-preserving affine transformations. This was first realized by von Neumann [31], who showed that a square is paradoxical using this expansion of the isometry group. In fact, it is sufficient to consider the group generated by  $SL_2(\mathbf{Z})$  and all translations; see [39]. In this section we discuss how the results of this paper are affected by considering linear, or affine, transformations instead of just isometries.

Let us consider the action of  $SL_2(\mathbf{R})$  on  $\mathbf{R}^2\setminus\{0\}$ . The two matrices,  $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$  and  $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$  freely generate a subgroup of  $SL_2(\mathbf{Z})$ , no nonidentity element of which has a fixed point in  $\mathbf{R}^2\setminus\{0\}$ ; this follows from the result of Magnus and Neumann mentioned in § 6, since an element of  $SL_2(\mathbf{Z})$  has a nonzero fixed point in  $\mathbf{R}^2$  if and only if it has trace 2. It follows by the technique of § 4 that  $SL_2(\mathbf{R})$  has a free subgroup with a perfect set of free generators whose action on  $\mathbf{R}^2\setminus\{0\}$  is fixed-point free. Therefore the action of  $SL_2(\mathbf{R})$  on  $\mathbf{R}^2\setminus\{0\}$  satisfies all the conclusions of Theorems 4 and 6.

Using techniques of functional analysis, J. Rosenblatt and R. Kallman (unpublished) have recently shown that the Lebesgue measurable subsets of  $\mathbb{R}^n\setminus\{0\}$   $(n\geq 2)$  do not bear a finitely additive,  $SL_n(\mathbb{Z})$ -invariant measure of total measure one. (For  $n\geq 3$  this uses the fact that  $SL_n(\mathbb{Z})$  has Kazhdan's Property T, while the  $\mathbb{R}^2$  case uses specific facts about representations of

 $SL_2(\mathbf{Z})$ ; see [41; Theorem 11.17].) Thus Theorem 9 does not extend to area-preserving affine transformations. It would be interesting if a paradoxical decomposition of  $\mathbf{R}^2 \setminus \{0\}$  using measurable sets, similar to the one illustrated in § 8, could be explicitly constructed. Some sort of paradoxical decomposition using measurable pieces must exist, by a general theorem of Tarski (see [41]), but it is not known if one using just four pieces exists. On the other hand, Belley and Prasad [4] have shown that there is a finitely additive measure on a certain (not too small) Boolean algebra of Borel subsets of  $\mathbf{R}^n$  that has total measure one and is invariant under all nonsingular affine transformations of  $\mathbf{R}^n$  (not just the measure-preserving ones).

Finally, we mention some unsolved problems about the existence of nice free groups of affine, area-preserving transformations, positive solutions to which would yield (via Theorems 4-6) paradoxical decompositions of  $\mathbf{R}^n$ . Let  $A_n(\mathbf{R})$  denote the group of affine transformations of  $\mathbf{R}^n$ , i.e., transformations of the form TL, where T is a translation and  $L \in GL_n(\mathbf{R})$ . Let  $SA_n(\mathbf{R})$  be the subgroup obtained by restricting L to  $SL_n(\mathbf{R})$ , and let  $SA_n(\mathbf{Z})$  consist of those TL where  $L \in SL_n(\mathbf{Z})$  and T is a translation by a vector in  $\mathbf{Z}^n$ . Note that  $SA_n(\mathbf{Z})$  acts on  $\mathbf{Z}^n$ . Since  $G(\mathbf{R}^3) \subseteq SA_3(\mathbf{R})$ , Theorem 1 yields that  $SA_3(\mathbf{R})$  has a free non-Abelian subgroup whose action on  $\mathbf{R}^3$  is fixed-point free. Consideration of  $\mathbf{Z}^3$  instead of  $\mathbf{R}^3$  leads to problem 1 below. Problem 2 is an attempt to get a version of these results for  $\mathbf{R}^2$  (rather than  $\mathbf{R}^2 \setminus \{0\}$ , which is treated at the beginning of this section). Only local commutativity is sought because of part (b) of the proposition below. Since  $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$  and its transpose freely generate a group of rank two, so do the two transformations:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence perhaps the subgroup of  $SA_2(\mathbf{Z})$  which these two transformations generate solves Problem 2 affirmatively. But we are unable to show that this subgroup is locally commutative.

Problems.

- 1. Does  $SA_3(\mathbf{Z})$  have a free subgroup of rank two which is fixed-point free on  $\mathbf{Z}^3$ ?
- 2. Does  $SA_2(\mathbf{R})$  (or  $SA_2(\mathbf{Z})$ ) have a subgroup of rank two which is locally commutative in its action on  $\mathbf{R}^2$  (or on  $\mathbf{Z}^2$ )?

Proposition 10.

- (a) If  $TL \in A_n(\mathbf{R})$  and TL has no fixed points in  $\mathbf{R}^n$ , then L has +1 as an eigenvalue, i.e., L has a fixed point in  $\mathbf{R}^n \setminus \{0\}$ .
- (b) If G is a subgroup of  $SA_2(\mathbf{R})$  which is fixed-point free on  $\mathbf{R}^2$  then G is solvable.

Proof.

- (a) Suppose T is a translation by the vector v. Since L(x) + v = x has no solution, the same is true of (L-I)(x) = -v, and therefore  $\det(L-I) = 0$ , i.e., 1 is an eigenvalue of L.
- (b) Let  $\sigma = TL$  and  $\tau = T'L'$  be in G. Then  $\sigma\tau = T''LL'$  so part (a) yields that each of L, L', LL' has 1 as an eigenvalue. Since these are  $2 \times 2$  matrices with determinant 1, this implies that all have trace 2. Hence, choosing an appropriate basis, we have  $L = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$  and  $L' = \begin{pmatrix} \alpha & \beta \\ \gamma & 2-\alpha \end{pmatrix}$ .

Then  $LL' = {\alpha + b\gamma \choose *}$ , and the trace of the latter being 2 yields that  $b\gamma = 0$ . But if either b or  $\gamma$  equal zero, then L and L' commute, which implies that the commutators  $\sigma\tau\sigma^{-1}\tau^{-1}$  and  $\sigma^{-1}\tau^{-1}\sigma\tau$  are pure translations. Hence [[G, G], [G, G]] is the identity subgroup, i.e., G is solvable.

Part (b) of the Proposition shows why there is no fixed-point free, non-Abelian free subgroup of  $SA_2(\mathbf{R})$ . But the following problem is unsolved.

*Problem 3.* Does there exist a free non-Abelian semigroup in  $SA_2(\mathbf{R})$  (or  $SA_2(\mathbf{Z})$ ) whose action on  $\mathbf{R}^2$  is fixed-point free?

Part (a) of Proposition 10 brings to light a distinction between the groups  $G(\mathbf{R}^n)$  according as n is even or odd. The proof of Theorem 1 for  $\mathbf{R}^3$  (§ 5) is essentially the same as the proof for  $S^{2n+1}$  given in § 4. Precisely, it is shown that  $A = \{\sigma \in G(\mathbf{R}^3) : \sigma \text{ has a fixed point in } \mathbf{R}^3\}$  is nowhere dense and, in fact, each  $R_w = f_w^{-1}(A)$  is nowhere dense in the appropriate product, where w is any group word in finitely many variables. While this is sufficient to get the existence of perfect free generating sets of fixed-point free subgroups in  $\mathbf{R}^3$  and beyond, the set A can fail to be nowhere dense in the higher dimensions. Indeed, consider  $\mathbf{R}^{2n}$ ,  $n \ge 1$ . Letting  $\pi: G(\mathbf{R}^{2n}) \to SO_{2n}$  be the canonical homomorphism, it follows from part (a) of Proposition 10 that  $G(\mathbf{R}^n) \setminus A \subseteq \pi^{-1}(B)$ , where  $B = \{L \in SO_{2n} : L \text{ has } 1 \text{ as an eigenvalue}\}$ . It is easy to see that B is nowhere dense and it follows that the same is true of  $\pi^{-1}(B)$ ; i.e., A has a nowhere dense complement. In odd

dimensions, however, the situation in  $\mathbb{R}^3$  is typical, as the following proposition shows.

PROPOSITION 11. If  $n \ge 1$  is odd then  $A = \{ \sigma \in G(\mathbf{R}^n) : \sigma \text{ has a fixed point in } \mathbf{R}^n \}$  is a nowhere dense subset of  $G(\mathbf{R}^n)$ .

Proof. It is an easy linear algebra exercise (generalizing Proposition 10 (a) above) to see that  $\sigma = TL$  has a fixed point in  $\mathbb{R}^n$  if and only if the translation vector of T is orthogonal to all vectors fixed by L. Since there is a basis for the fixed space of L that consists of vectors whose entries are polynomials in the entries of L (Gaussian elimination and scaling), this latter condition on TL is equivalent to the vanishing of a polynomial in the entries of  $\sigma$ . But the condition is not universally true in  $G(\mathbb{R}^n)$  since any pure translation has no fixed points; therefore the technique introduced in § 4 implies that A is nowhere dense, as desired.

This proposition, in exactly the same cases, is valid for  $SO_{n+1}$ 's action on  $S^n$  (see § 4). The following extension of these results is a refinement of the theorems on the existence of free, fixed-point free groups of isometries of rank m: it shows that in these cases almost all (from the category point of view) m-tuples of isometries are free generators of fixed-point free groups of isometries.

PROPOSITION 12. Suppose n is odd and  $n \ge 3$ , and X is one of  $\mathbf{R}^n$  or  $S^n$ . Then any m elements of G(X), with the exception of a meager set in  $G(X)^m$ , are free generators of a fixed-point free subgroup of G(X).

*Proof.* For the spherical case this follows from § 4, where it was shown that  $\bigcup \{R_w : w \text{ is a group word in } m \text{ variables} \}$  is comeager. The Euclidean case is proved by observing (see Proposition 11's proof and § 5) that there is a function p that is a polynomial in the entries of  $\sigma_1, ..., \sigma_m$  such that p = 0 if and only if  $f_w(\sigma_1, ..., \sigma_m) \in A$ . Since, by the rank two case of Theorem 1 (a), f is not identically zero,  $f_w^{-1}(A)$  is nowhere dense. Therefore the union over all words in m variables is meager, as desired.