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$f_w(SO^m_{n+1})$ is not contained in a proper algebraic subset (in this case, A) of SO_{n+1}. This completes the proof of Theorem 1 (a) for S^n.

Next, consider Theorem 1 (c) for S^n. First observe that this can be proved for SO_3 by the technique above, if A is taken to consist simply of the identity. This is because the action of SO_3 on S^2 is locally commutative, so all that is needed is a perfect set of free generators, which in turn requires only that each R_w be nowhere dense. Theorem 1 of [5] again applies, because A is an algebraic set: membership in A is equivalent to the simultaneous vanishing of $(n+1)^2$ polynomials which, by using a sum of squares, is equivalent to the vanishing of a single polynomial. For higher dimensions, we appeal to the technique used by Borel to get locally commutative free subgroups of SO_{n+1}. In [5, p. 162] he showed that, if $n \geqslant 2$, SO_3 may be represented as a subgroup H of SO_{n+1} where H's action on S^n is locally commutative. Hence the perfect free generating set in SO_3 yields a perfect subset of H which is the desired free generating set in SO_{n+1}.

§ 5. Euclidean Spaces

For the Euclidean case of Theorem 1, it suffices to consider $\mathbf{R}^3$, since any isometry of $\mathbf{R}^3$ can be extended to one in higher dimensions by simply fixing the additional coordinates; this introduces no new fixed points. Now, $\mathbf{R}^3$ can be handled in a way entirely similar to S^n. Any orientation-preserving isometry of $\mathbf{R}^3$ is a screw-motion, i.e. a rotation $\rho \in SO_3$ followed by a translation τ. Such isometries may be represented as elements of $SL_4(\mathbf{R})$ as follows: if $\sigma = \tau\rho$ where ρ corresponds to $(a_{ij}) \in SO_3$ and τ is a translation by (v_1, v_2, v_3), then identify σ with the matrix

$$\begin{pmatrix} & & & v_1 \\ & a_{ij} & & v_2 \\ & & & v_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since composition of isometries corresponds to matrix multiplication, this shows that $G(\mathbf{R}^3)$ may be viewed as a connected (6-dimensional) analytic submanifold of $\mathbf{R}^{12}$. Now, the proof can proceed exactly as for spheres, once it is shown that the existence of a fixed point is equivalent to the vanishing of a polynomial. But a screw-motion σ has a fixed point if and only if the translation vector is perpendicular to the axis of the rotation. Since the axis of a rotation $(a_{ij}) \in SO_3$ is parallel to $(a_{32}-a_{23}, a_{13}-a_{31}, a_{21}-a_{12})$, σ has a fixed point if and only if $v_1(a_{32}-a_{23}) + v_2(a_{13}-a_{31}) + v_3(a_{21}-a_{12}) = 0$. This completes the proof of Theorem 1 (a) for $\mathbf{R}^n$.

§ 6. Hyperbolic Spaces

Here we meet a case where the existence of a free, fixed-point free group of isometries having rank 2 does not imply the existence of such a group having uncountable rank. The hyperbolic plane is such a space.

If H^2 is identified with the upper half-plane of $\mathbf{C}$, then $G(H^2)$ corresponds to linear fractional transformations $z \mapsto \dfrac{az + b}{cz + d}$, where a, b, c, d are real and $ad - bc \neq 0$. Since it may be assumed that $ad - bc = 1$, this group is isomorphic to $PSL_2(\mathbf{R})$. A nonidentity element of $PSL_2(\mathbf{R})$ is called elliptic, parabolic, or hyperbolic according as the absolute value of its trace is less than, equal to, or greater than two; the nonidentity elements of $G(H^2)$ with a fixed point in H^2 correspond to the elliptic elements of $PSL_2(\mathbf{R})$. See [18] for more details about this interpretation of $PSL_2(\mathbf{R})$. The following theorem clarifies the situation regarding fixed-point free subgroups of $G(H^2)$.

Theorem 3. (Siegel) *If F is a free subgroup of $PSL_2(\mathbf{R})$ then F is discrete if and only if F has no elliptic elements.*

Theorem 3 is a rephrasing of the result of [34] (see also [15]). An elementary proof appears in [41]. The forward direction is an immediate consequence of the fact that the nondiscrete cyclic subgroups of $PSL_2(\mathbf{R})$ are precisely the ones generated by an elliptic element of infinite order. This fact also yields the reverse direction in the case when F is cyclic. Siegel gave an algebraic proof of the reverse direction for noncyclic free groups. This can also be obtained by first using techniques of Lie algebras to show that a nondiscrete, nonsolvable subgroup of $PSL_2(\mathbf{R})$ is dense in $PSL_2(\mathbf{R})$, and observing that the elliptics form an open set; this approach is due, independently, to A. Borel and D. Sullivan.

The forward (easy) direction of Theorem 3 yields a proof of the positive part of Theorem 1 (b) for H^2 (and hence for H^n, $n \geqslant 2$), since it implies that a discrete free group of rank two has no elliptic elements. Therefore