

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 30 (1984)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LARGE FREE GROUPS OF ISOMETRIES AND THEIR GEOMETRICAL USES
Autor: Mycielski, Jan / Wagon, Stan
Kapitel: §5. Euclidean Spaces
DOI: <https://doi.org/10.5169/seals-53829>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

$f_w(SO_{n+1}^m)$ is not contained in a proper algebraic subset (in this case, A) of SO_{n+1} . This completes the proof of Theorem 1 (a) for S^n .

Next, consider Theorem 1 (c) for S^n . First observe that this can be proved for SO_3 by the technique above, if A is taken to consist simply of the identity. This is because the action of SO_3 on S^2 is locally commutative, so all that is needed is a perfect set of free generators, which in turn requires only that each R_w be nowhere dense. Theorem 1 of [5] again applies, because A is an algebraic set: membership in A is equivalent to the simultaneous vanishing of $(n+1)^2$ polynomials which, by using a sum of squares, is equivalent to the vanishing of a single polynomial. For higher dimensions, we appeal to the technique used by Borel to get locally commutative free subgroups of SO_{n+1} . In [5, p. 162] he showed that, if $n \geq 2$, SO_3 may be represented as a subgroup H of SO_{n+1} where H 's action on S^n is locally commutative. Hence the perfect free generating set in SO_3 yields a perfect subset of H which is the desired free generating set in SO_{n+1} .

§ 5. EUCLIDEAN SPACES

For the Euclidean case of Theorem 1, it suffices to consider \mathbf{R}^3 , since any isometry of \mathbf{R}^3 can be extended to one in higher dimensions by simply fixing the additional coordinates; this introduces no new fixed points. Now, \mathbf{R}^3 can be handled in a way entirely similar to S^n . Any orientation-preserving isometry of \mathbf{R}^3 is a screw-motion, i.e. a rotation $\rho \in SO_3$ followed by a translation τ . Such isometries may be represented as elements of $SL_4(\mathbf{R})$ as follows: if $\sigma = \tau\rho$ where ρ corresponds to $(a_{ij}) \in SO_3$ and τ is a translation by (v_1, v_2, v_3) , then identify σ with the matrix

$$\begin{pmatrix} & & & v_1 \\ & a_{ij} & & v_2 \\ & & & v_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since composition of isometries corresponds to matrix multiplication, this shows that $G(\mathbf{R}^3)$ may be viewed as a connected (6-dimensional) analytic submanifold of \mathbf{R}^{12} . Now, the proof can proceed exactly as for spheres, once it is shown that the existence of a fixed point is equivalent to the

vanishing of a polynomial. But a screw-motion σ has a fixed point if and only if the translation vector is perpendicular to the axis of the rotation. Since the axis of a rotation $(a_{ij}) \in SO_3$ is parallel to $(a_{32} - a_{23}, a_{13} - a_{31}, a_{21} - a_{12})$, σ has a fixed point if and only if $v_1(a_{32} - a_{23}) + v_2(a_{13} - a_{31}) + v_3(a_{21} - a_{12}) = 0$. This completes the proof of Theorem 1 (a) for \mathbf{R}^n .

§ 6. HYPERBOLIC SPACES

Here we meet a case where the existence of a free, fixed-point free group of isometries having rank 2 does not imply the existence of such a group having uncountable rank. The hyperbolic plane is such a space.

If H^2 is identified with the upper half-plane of \mathbf{C} , then $G(H^2)$ corresponds to linear fractional transformations $z \mapsto \frac{az + b}{cz + d}$, where a, b, c, d are real and $ad - bc \neq 0$. Since it may be assumed that $ad - bc = 1$, this group is isomorphic to $PSL_2(\mathbf{R})$. A nonidentity element of $PSL_2(\mathbf{R})$ is called elliptic, parabolic, or hyperbolic according as the absolute value of its trace is less than, equal to, or greater than two; the nonidentity elements of $G(H^2)$ with a fixed point in H^2 correspond to the elliptic elements of $PSL_2(\mathbf{R})$. See [18] for more details about this interpretation of $PSL_2(\mathbf{R})$. The following theorem clarifies the situation regarding fixed-point free subgroups of $G(H^2)$.

THEOREM 3. (Siegel) *If F is a free subgroup of $PSL_2(\mathbf{R})$ then F is discrete if and only if F has no elliptic elements.*

Theorem 3 is a rephrasing of the result of [34] (see also [15]). An elementary proof appears in [41]. The forward direction is an immediate consequence of the fact that the nondiscrete cyclic subgroups of $PSL_2(\mathbf{R})$ are precisely the ones generated by an elliptic element of infinite order. This fact also yields the reverse direction in the case when F is cyclic. Siegel gave an algebraic proof of the reverse direction for noncyclic free groups. This can also be obtained by first using techniques of Lie algebras to show that a nondiscrete, nonsolvable subgroup of $PSL_2(\mathbf{R})$ is dense in $PSL_2(\mathbf{R})$, and observing that the elliptics form an open set; this approach is due, independently, to A. Borel and D. Sullivan.

The forward (easy) direction of Theorem 3 yields a proof of the positive part of Theorem 1 (b) for H^2 (and hence for H^n , $n \geq 2$), since it implies that a discrete free group of rank two has no elliptic elements. Therefore