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$f_w(SO_{n+1}^m)$  is not contained in a proper algebraic subset (in this case,  $A$ ) of  $SO_{n+1}$ . This completes the proof of Theorem 1 (a) for  $S^n$ .

Next, consider Theorem 1 (c) for  $S^n$ . First observe that this can be proved for  $SO_3$  by the technique above, if  $A$  is taken to consist simply of the identity. This is because the action of  $SO_3$  on  $S^2$  is locally commutative, so all that is needed is a perfect set of free generators, which in turn requires only that each  $R_w$  be nowhere dense. Theorem 1 of [5] again applies, because  $A$  is an algebraic set: membership in  $A$  is equivalent to the simultaneous vanishing of  $(n+1)^2$  polynomials which, by using a sum of squares, is equivalent to the vanishing of a single polynomial. For higher dimensions, we appeal to the technique used by Borel to get locally commutative free subgroups of  $SO_{n+1}$ . In [5, p. 162] he showed that, if  $n \geq 2$ ,  $SO_3$  may be represented as a subgroup  $H$  of  $SO_{n+1}$  where  $H$ 's action on  $S^n$  is locally commutative. Hence the perfect free generating set in  $SO_3$  yields a perfect subset of  $H$  which is the desired free generating set in  $SO_{n+1}$ .

### § 5. EUCLIDEAN SPACES

For the Euclidean case of Theorem 1, it suffices to consider  $\mathbf{R}^3$ , since any isometry of  $\mathbf{R}^3$  can be extended to one in higher dimensions by simply fixing the additional coordinates; this introduces no new fixed points. Now,  $\mathbf{R}^3$  can be handled in a way entirely similar to  $S^n$ . Any orientation-preserving isometry of  $\mathbf{R}^3$  is a screw-motion, i.e. a rotation  $\rho \in SO_3$  followed by a translation  $\tau$ . Such isometries may be represented as elements of  $SL_4(\mathbf{R})$  as follows: if  $\sigma = \tau\rho$  where  $\rho$  corresponds to  $(a_{ij}) \in SO_3$  and  $\tau$  is a translation by  $(v_1, v_2, v_3)$ , then identify  $\sigma$  with the matrix

$$\begin{pmatrix} & & & v_1 \\ & & & v_2 \\ & a_{ij} & & v_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since composition of isometries corresponds to matrix multiplication, this shows that  $G(\mathbf{R}^3)$  may be viewed as a connected (6-dimensional) analytic submanifold of  $\mathbf{R}^{12}$ . Now, the proof can proceed exactly as for spheres, once it is shown that the existence of a fixed point is equivalent to the

vanishing of a polynomial. But a screw-motion  $\sigma$  has a fixed point if and only if the translation vector is perpendicular to the axis of the rotation. Since the axis of a rotation  $(a_{ij}) \in SO_3$  is parallel to  $(a_{32} - a_{23}, a_{13} - a_{31}, a_{21} - a_{12})$ ,  $\sigma$  has a fixed point if and only if  $v_1(a_{32} - a_{23}) + v_2(a_{13} - a_{31}) + v_3(a_{21} - a_{12}) = 0$ . This completes the proof of Theorem 1 (a) for  $\mathbf{R}^n$ .

## § 6. HYPERBOLIC SPACES

Here we meet a case where the existence of a free, fixed-point free group of isometries having rank 2 does not imply the existence of such a group having uncountable rank. The hyperbolic plane is such a space.

If  $H^2$  is identified with the upper half-plane of  $\mathbf{C}$ , then  $G(H^2)$  corresponds to linear fractional transformations  $z \mapsto \frac{az + b}{cz + d}$ , where  $a, b, c, d$  are real and  $ad - bc \neq 0$ . Since it may be assumed that  $ad - bc = 1$ , this group is isomorphic to  $PSL_2(\mathbf{R})$ . A nonidentity element of  $PSL_2(\mathbf{R})$  is called elliptic, parabolic, or hyperbolic according as the absolute value of its trace is less than, equal to, or greater than two; the nonidentity elements of  $G(H^2)$  with a fixed point in  $H^2$  correspond to the elliptic elements of  $PSL_2(\mathbf{R})$ . See [18] for more details about this interpretation of  $PSL_2(\mathbf{R})$ . The following theorem clarifies the situation regarding fixed-point free subgroups of  $G(H^2)$ .

**THEOREM 3.** (Siegel) *If  $F$  is a free subgroup of  $PSL_2(\mathbf{R})$  then  $F$  is discrete if and only if  $F$  has no elliptic elements.*

Theorem 3 is a rephrasing of the result of [34] (see also [15]). An elementary proof appears in [41]. The forward direction is an immediate consequence of the fact that the nondiscrete cyclic subgroups of  $PSL_2(\mathbf{R})$  are precisely the ones generated by an elliptic element of infinite order. This fact also yields the reverse direction in the case when  $F$  is cyclic. Siegel gave an algebraic proof of the reverse direction for noncyclic free groups. This can also be obtained by first using techniques of Lie algebras to show that a nondiscrete, nonsolvable subgroup of  $PSL_2(\mathbf{R})$  is dense in  $PSL_2(\mathbf{R})$ , and observing that the elliptics form an open set; this approach is due, independently, to A. Borel and D. Sullivan.

The forward (easy) direction of Theorem 3 yields a proof of the positive part of Theorem 1 (b) for  $H^2$  (and hence for  $H^n$ ,  $n \geq 2$ ), since it implies that a discrete free group of rank two has no elliptic elements. Therefore