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LARGE FREE GROUPS OF ISOMETRIES AND THEIR GEOMETRICAL USES

by Jan MYCIELSKI and Stan WAGON

§ 1. INTRODUCTION

Various geometric constructions related to the Banach-Tarski paradoxical decomposition of the sphere require the existence of free groups of isometries acting without fixed points (i.e., no nonidentity element of the group has a fixed point) or in a locally commutative way (i.e., if two group elements have a common fixed point, then they commute). For some of these constructions a free group of rank 2 is sufficient; others require one of rank $2^{\aleph_0}$. It is the purpose of this paper to fill a few gaps in this subject, where the underlying spaces are the spheres S^n, the Euclidean spaces $\mathbf{R}^n$, and the hyperbolic spaces H^n.

For groups of rank 2, all cases of this problem have been solved, and we shall review these results in § 2. For the case of rank $2^{\aleph_0}$, we present a unified approach (§ 4-6) to the known results which is sufficiently general to settle the heretofore unresolved cases, H^3, S^{4n+1} and S^4. The main idea of our proofs is a general topological technique (introduced in [25]) that uses the groups of rank 2 to obtain a perfect (i.e., closed and without isolated points) set of free generators. In all cases except H^2, the existence of a fixed-point free or locally commutative rank 2 free group of isometries implies the existence of a group of rank $2^{\aleph_0}$ with the same properties.

In § 7 and § 8 we discuss the geometric consequences of the existence of large free groups of isometries. For example, each of S^n $(n \geqslant 2)$, H^n $(n \geqslant 2)$ and $\mathbf{R}^n$ $(n \geqslant 3)$ contains a set which is, simultaneously, a third, a quarter, ..., a $2^{\aleph_0}$ th part of the space. In § 8 we show how paradoxical decompositions of H^n $(n \geqslant 2)$ can be constructed using Borel sets (and not using the Axiom of Choice). However, such paradoxical decompositions of $\mathbf{R}^n$ either do not exist, even allowing arbitrary sets ($n=1$ or 2), or exist ($n \geqslant 3$), but require nonmeasurable sets and the Axiom of Choice.

Finally, in § 9 we discuss what can be done in $\mathbf{R}^2$ if we allow area-preserving linear or affine transformations instead of just isometries.

Proofs of several of the results mentioned or used in this paper, such as Theorems 3, 4 (b) and (c), 5, 6, and 7, may be found in [41].

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§ 2. The Main Theorem

The action of a group, G, on a set, X, is called *fixed-point free* if $g(x) \neq x$ for all $x \in X$ and $g \in G \backslash \{I\}$ (I is the identity of G). The action is called *locally commutative* if, for each $x \in X$, $\{\sigma \in G : \sigma(x) = x\}$ is a commutative subgroup of G; equivalently, if two elements of G have a common fixed point in X then they commute. For any group G and any abstract (reduced) group word w in m variables, the function $f_w : G^m \to G$ is defined by $f_w(\sigma_1, ..., \sigma_m) = w(\sigma_1, ..., \sigma_m)$.

If X is a metric space and also an oriented manifold, then $G(X)$ denotes the group of orientation-preserving isometries of X, with its natural topology. In particular, $G(S^n) = SO_{n+1}$, $G(H^2) = PSL_2(\mathbf{R})$ and $G(H^3) = PSL_2(\mathbf{C})$.

A set in a complete metric space is called *perfect* if it is nonempty, closed and without isolated points; a perfect set has at least $2^{\aleph_0}$ elements.

Theorem 1.

(a) *Each of the groups* $G(S^n)$, *where* n *is odd and* $n \geqslant 2$, $G(\mathbf{R}^n)$, *where* $n \geqslant 3$, *and* $G(H^n)$, *where* $n \geqslant 3$, *has a free subgroup with a perfect set of free generators whose action on the space is fixed-point free.*

(b) $G(H^2)$ *has a discrete free subgroup of rank* 2 *(and hence also rank* $\aleph_0$*) which is fixed-point free, but no such free subgroup of* $G(H^2)$ *can have uncountable rank.*

(c) $G(H^2)$ *and each of the groups* $G(S^n)$, $n \geqslant 2$, *have locally commutative free subgroups with a perfect set of free generators.*

The above theorem is false in all omitted dimensions. This is because the isometry groups in the low dimensions are all solvable, and hence contain no free subgroup of rank 2. Also, each element of SO_{2n+1} has a fixed point on S^{2n}, and this is why part (a) fails for spheres of even

dimension. Of course, an uncountable subgroup of $G(H^n)$ (or any infinite subgroup of the compact group SO_n) cannot be discrete. Moreover, $G(\mathbf{R}^n)$ has the Abelian (and therefore amenable) group of translations as a closed normal subgroup, and the quotient is the compact (and therefore amenable as a topological group) group SO_n. It follows that $G(\mathbf{R}^n)$ is an amenable topological group, whence it has no discrete free subgroup of rank two (see [12, § 2]).

The results of Theorem 1 are known except for part (a) for H^3 and S^n if $n \equiv 1 \pmod 4$, and part (c) for S^4. The history of the known cases is the following. The earliest results on free isometry groups are due to Klein and Fricke [16] and Hausdorff [14]. The former showed that $PSL_2(\mathbf{Z})$ is isomorphic to the free product $\mathbf{Z}_2 * \mathbf{Z}_3$, (see [17, Appendix]), whence $PSL_2(\mathbf{R})$, which is isomorphic to $G(H^2)$, contains a free subgroup of rank 2. Since the entire action of $PSL_2(\mathbf{R})$ on H^2 is locally commutative (see § 6), this yields part (c) for H^2 and rank 2. Hausdorff showed that $\mathbf{Z}_2 * \mathbf{Z}_3$ also appears as a subgroup of SO_3. Again, the action of the rotation group SO_3 on S^2 is locally commutative, so this yields part (c) for S^2 and rank 2. This was the foundation of Hausdorff's theorem that there is no finitely additive, rotation-invariant measure defined for all subsets of S^2 and having total measure one. It also forms the basis of the Banach-Tarski paradoxical decomposition of a sphere. It was not until much later, however, that the advantages of local commutativity for such constructions were recognized [34, 13]. The simplest proof that $\mathbf{Z}_2 * \mathbf{Z}_3$ embeds in SO_3 may be found in [33]. However, see [10] for a beautiful proof using tetrahedra that $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$ (and hence a free non-Abelian group) embeds in the group of isometries of $\mathbf{R}^3$. For the general problem of the existence of free subgroups of topological groups see the literature quoted in [28, p. 681].

Part (c) for S^2 was first proved by Sierpiński [36]. Further results on the embedding of free products into SO_3 and $SL_2(\mathbf{R})$ are due to Balcerzyk and Mycielski [2]; see also Nisnewitsch [32]. Dekker [7, 8] made an extensive investigation into higher dimensions and the non-Euclidean cases, proving part (c) for H^2 and S^n (except for the case of S^4) and part (a) for S^n provided $n \equiv -1 \pmod 4$. Part (a) for $\mathbf{R}^n$ was proved by Dekker [9] and, independently, by Mycielski and Świerczkowski [29]. The remaining cases of parts (a) and (c) for groups of rank 2 were proved recently by Deligne and Sullivan [11] (part (a) for S^n, $n \equiv 1 \pmod 4$) and Borel [5] (part (c) for S^4). The positive result in part (b) is a consequence of well-known facts about $PSL_2(\mathbf{R})$ (first pointed out in [5]), while the negative result is a consequence of a theorem on $PSL_2(\mathbf{R})$ due to Siegel [35] (see § 6).

The proof of Theorem 1 to be given below will assume all the aforementioned results about the existence of free groups of rank 2. An important fact is that a free group of rank 2 has a free subgroup of rank $\aleph_0$: if σ, τ freely generate the group, F, of rank 2 then $\{\sigma^i \tau \sigma^{-i} : i = 0, 1, 2, \ldots\}$ is a set of free generators of a subgroup of F, and the same is true of $\{\sigma^i \tau^i : i = 1, 2, \ldots\}$. More generally (see [21, p. 195]) a free product $A * B$ must have a free subgroup of rank $\aleph_0$ unless A or B is a one-element group or $A \cong \mathbf{Z}_2 \cong B$.

We shall also consider elliptic spaces, L^n, which are represented by S^n, with antipodal points identified. Hence the isometry group of L^n is SO_{n+1}, if n is even, or $SO_{n+1}/\{\pm I\}$, if n is odd. Note that any fixed-point free or locally commutative free subgroup of $G(S^n)$ induces such a subgroup of L^n's isometry group. If a nontrivial (reduced) word w became the identity or gained a fixed point when viewed as acting on L^n, then w^2 would be the identity or have a fixed point as a member of SO_{n+1}. Furthermore, if two words, u and v, share a fixed point on L^n then u^2 and v^2 share a fixed point on S^n. Since, in a free group, u and v commute if and only if u^2 and v^2 do ([21, p. 41]), this shows that a locally commutative free subgroup of SO_{n+1} induces one of the same rank in L^n's isometry group.

§ 3. A Preliminary Theorem About Metric Spaces

The passage from a free group of rank 2 to one of rank $2^{\aleph_0}$ with the same fixed-point properties utilizes the following general theorem of Mycielski [25].

THEOREM 2. *Let X be a complete, separable metric space with no isolated points and suppose that, for each $i < \infty$, R_i is a nowhere dense subset of some finite product X^{r_i}. Then there is a perfect subset F of X which avoids each R_i in the sense that any r_i-tuple of distinct elements of F does not lie in R_i.*

The proof of this theorem is not difficult: one constructs a tree of open sets such that no sequence from distinct nodes at level m lies in any R_i with $i \leqslant m$. Then, provided the open sets are small enough (precisely, their diameters converge to zero along branches, and the closure of any node is contained in one of the open sets at the previous level), F may be obtained as the collection of points lying in the intersections along infinite branches of the tree.

For various applications and generalizations of this result, see [25, 26]. The separability condition is not essential, but its presence allows the proof to be carried out without using the Axiom of Choice.

Our applications of this theorem will involve finding appropriate collections of algebraic or analytic surfaces R_i, such that the set F that avoids them will be the desired set of free generators. The fact that the desired free group of rank two (and hence $\aleph_0$) exists will be used to verify that each R_i is indeed nowhere dense.

§ 4. Spheres

First we prove Theorem 1 (a) for S^n(n odd, $n \geqslant 3$). Let $A = \{\sigma \in SO_{n+1} : \sigma$ has a fixed point in $S^n\}$; therefore $\sigma \in A$ if and only if $\det(\sigma - I) = 0$. For each nonidentity reduced group word w in m variables, let $R_w = f_w^{-1}(A)$; thus $(\sigma_1, ..., \sigma_m) \in R_w$ if and only if $w(\sigma_1, ..., \sigma_m)$ has a fixed point. It is enough to show that each R_w is nowhere dense, for then Theorem 2 may be applied to the countable set of relations $\{R_w\}$ to get a perfect set $F \subseteq SO_{n+1}$. Since F avoids each R_w, no word using elements of F has a fixed point on S^n. This implies, in particular, that no such word equals the identity, and so F is the desired set of rotations.

To see that each R_w is nowhere dense, we view $SO(n+1)^m$ as a (connected) analytic submanifold of $\mathbf{R}^{m(n+1)^2}$ (of dimension $\frac{1}{2}n(n+1)m$). We need an analytic $f : SO_{n+1}^m \to \mathbf{R}$ such that $R_w = f^{-1}(\{0\})$. Such a function exists because membership in R_w is equivalent to the condition that $+1$ is an eigenvalue of w. Hence we may simply let

$$f(\sigma_1, ..., \sigma_m) = \det(w(\sigma_1, ..., \sigma_m) - I).$$

Since f is a polynomial in the $m(n+1)^2$ entries of the σ_i (this uses the fact that $\det(\sigma_i) = 1$ to obtain that each entry of σ_i^{-1} is a polynomial in the entries of σ_i), f is analytic on SO_{n+1}^m.

Since f is continuous, R_w is closed, so it remains to show that R_w's interior is empty. Suppose not. Since SO_{n+1}^m is connected, an analytic function that vanishes on a nonempty open set must vanish everywhere. Hence $R_w = SO_{n+1}^m$, which contradicts the existence of a free subgroup of SO_{n+1} of rank m which is fixed-point free (which was proved in [7, 11]). Alternatively, $R_w = SO_{n+1}^m$ contradicts Theorem 1 of [5] which asserts that

$f_w(SO^m_{n+1})$ is not contained in a proper algebraic subset (in this case, A) of SO_{n+1}. This completes the proof of Theorem 1 (a) for S^n.

Next, consider Theorem 1 (c) for S^n. First observe that this can be proved for SO_3 by the technique above, if A is taken to consist simply of the identity. This is because the action of SO_3 on S^2 is locally commutative, so all that is needed is a perfect set of free generators, which in turn requires only that each R_w be nowhere dense. Theorem 1 of [5] again applies, because A is an algebraic set: membership in A is equivalent to the simultaneous vanishing of $(n+1)^2$ polynomials which, by using a sum of squares, is equivalent to the vanishing of a single polynomial. For higher dimensions, we appeal to the technique used by Borel to get locally commutative free subgroups of SO_{n+1}. In [5, p. 162] he showed that, if $n \geqslant 2$, SO_3 may be represented as a subgroup H of SO_{n+1} where H's action on S^n is locally commutative. Hence the perfect free generating set in SO_3 yields a perfect subset of H which is the desired free generating set in SO_{n+1}.

§ 5. Euclidean Spaces

For the Euclidean case of Theorem 1, it suffices to consider $\mathbf{R}^3$, since any isometry of $\mathbf{R}^3$ can be extended to one in higher dimensions by simply fixing the additional coordinates; this introduces no new fixed points. Now, $\mathbf{R}^3$ can be handled in a way entirely similar to S^n. Any orientation-preserving isometry of $\mathbf{R}^3$ is a screw-motion, i.e. a rotation $\rho \in SO_3$ followed by a translation τ. Such isometries may be represented as elements of $SL_4(\mathbf{R})$ as follows: if $\sigma = \tau\rho$ where ρ corresponds to $(a_{ij}) \in SO_3$ and τ is a translation by (v_1, v_2, v_3), then identify σ with the matrix

$$\begin{pmatrix} & & & v_1 \\ & a_{ij} & & v_2 \\ & & & v_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Since composition of isometries corresponds to matrix multiplication, this shows that $G(\mathbf{R}^3)$ may be viewed as a connected (6-dimensional) analytic submanifold of $\mathbf{R}^{12}$. Now, the proof can proceed exactly as for spheres, once it is shown that the existence of a fixed point is equivalent to the

vanishing of a polynomial. But a screw-motion σ has a fixed point if and only if the translation vector is perpendicular to the axis of the rotation. Since the axis of a rotation $(a_{ij}) \in SO_3$ is parallel to $(a_{32}-a_{23}, a_{13}-a_{31}, a_{21}-a_{12})$, σ has a fixed point if and only if $v_1(a_{32}-a_{23}) + v_2(a_{13}-a_{31}) + v_3(a_{21}-a_{12}) = 0$. This completes the proof of Theorem 1 (a) for $\mathbf{R}^n$.

§ 6. Hyperbolic Spaces

Here we meet a case where the existence of a free, fixed-point free group of isometries having rank 2 does not imply the existence of such a group having uncountable rank. The hyperbolic plane is such a space.

If H^2 is identified with the upper half-plane of $\mathbf{C}$, then $G(H^2)$ corresponds to linear fractional transformations $z \mapsto \dfrac{az+b}{cz+d}$, where a, b, c, d are real and $ad - bc \neq 0$. Since it may be assumed that $ad - bc = 1$, this group is isomorphic to $PSL_2(\mathbf{R})$. A nonidentity element of $PSL_2(\mathbf{R})$ is called elliptic, parabolic, or hyperbolic according as the absolute value of its trace is less than, equal to, or greater than two; the nonidentity elements of $G(H^2)$ with a fixed point in H^2 correspond to the elliptic elements of $PSL_2(\mathbf{R})$. See [18] for more details about this interpretation of $PSL_2(\mathbf{R})$. The following theorem clarifies the situation regarding fixed-point free subgroups of $G(H^2)$.

THEOREM 3. (Siegel) *If F is a free subgroup of $PSL_2(\mathbf{R})$ then F is discrete if and only if F has no elliptic elements.*

Theorem 3 is a rephrasing of the result of [34] (see also [15]). An elementary proof appears in [41]. The forward direction is an immediate consequence of the fact that the nondiscrete cyclic subgroups of $PSL_2(\mathbf{R})$ are precisely the ones generated by an elliptic element of infinite order. This fact also yields the reverse direction in the case when F is cyclic. Siegel gave an algebraic proof of the reverse direction for noncyclic free groups. This can also be obtained by first using techniques of Lie algebras to show that a nondiscrete, nonsolvable subgroup of $PSL_2(\mathbf{R})$ is dense in $PSL_2(\mathbf{R})$, and observing that the elliptics form an open set; this approach is due, independently, to A. Borel and D. Sullivan.

The forward (easy) direction of Theorem 3 yields a proof of the positive part of Theorem 1 (b) for H^2 (and hence for H^n, $n \geqslant 2$), since it implies that a discrete free group of rank two has no elliptic elements. Therefore

any rank-two free subgroup of $SL_2(\mathbf{Z})$ is fixed-point free when viewed as a group of isometries of H^2. The simplest example of such a free group is the subgroup generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ (see [6]). (Recall also the result, mentioned in § 2, that $PSL_2(\mathbf{Z}) \cong \mathbf{Z}_2 * \mathbf{Z}_3$.)

Moreover, the reverse direction of Theorem 3 yields the negative part of Theorem 1 (b). For an uncountable subgroup of $PSL_2(\mathbf{R})$ is necessarily nondiscrete, and so an uncountable free subgroup must contain an elliptic element.

Let us point out why the perfect set technique of the previous sections breaks down in H^2. A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbf{R})$ is elliptic if and only if $(a+d)^2 < 4$, which is a polynomial inequality rather than an equality. Therefore the elliptics do not form a closed set, and hence they cannot be the zero set of an analytic function.

But the method of § 4 easily yields Theorem 1 (c) for H^2. Simply let $R_w = f_w^{-1}(\{I\})$; R_w is the zero set of an analytic function. Therefore the method of § 4 yields a free subgroup of $SL_2(\mathbf{R})$ (and hence of $PSL_2(\mathbf{R})$) with a perfect set of free generators. This proves Theorem 1 (c) for H^2, since the entire action of $PSL_2(\mathbf{R})$ on H^2 is locally commutative: if two elliptics share a fixed point, then they have the same set of fixed points in $\mathbf{C} \cup \{\infty\}$, so they commute.

A large, free locally commutative subgroup of $G(H^2)$ immediately yields such a subgroup of $G(H^n)$, $n \geqslant 3$, but a stronger result, namely Theorem 1 (a), is true in these higher dimensions. Consider first the case $n \geqslant 4$. By considering H^4 as the upper half-space in $\mathbf{R}^4$, it is easy to see that there is a monomorphism of $G(\mathbf{R}^3)$ into $G(H^4)$; any isometry of $\mathbf{R}^3$ is extended to H^4 by fixing the additional coordinate. Since a fixed-point free isometry remains so, Theorem 1 (a) for H^4 (and hence for H^n, $n \geqslant 4$) is a consequence of the corresponding result for $\mathbf{R}^3$. This method fails in H^3 however, since $G(\mathbf{R}^2)$ has no non-Abelian free subgroup.

To prove Theorem 1 (a) for H^3, we shall use the facts (see [3]) that $G(H^3)$ is isomorphic to $PSL_2(\mathbf{C})$ and that the elliptic transformations have real trace. (The elliptics, i.e., those nonidentity transformations in $PSL_2(\mathbf{C})$ fixing a point in H^3, are precisely the transformations whose trace is real and lies in the open interval $(-2, 2)$.) It will be more convenient to work in $SL_2(\mathbf{C})$ and there is no loss in so doing, since a free subgroup of $SL_2(\mathbf{C})$ induces one in $PSL_2(\mathbf{C})$. As before, consider a word w in m variables, and define R_w to be $\{(\sigma_1, ..., \sigma_m) \in SL_2(\mathbf{C})^m : w(\sigma_1, ..., \sigma_m) \text{ is elliptic}\}$. We wish

to show that R_w is nowhere dense, and to this end we consider the superset R_w^* of R_w defined by:

$$R_w^* = \{(\sigma_1, ..., \sigma_m) \in SL_2(\mathbf{C})^m : \operatorname{trace}(w(\sigma_1, ..., \sigma_m)) \in \mathbf{R}\}.$$

LEMMA. *R_w^* is a nowhere dense subset of $SL_2(\mathbf{C})^m$.*

Proof. We shall view $SL_2(\mathbf{C})^m$ as a connected real analytic submanifold of $\mathbf{R}^{8m}$. If $a_1, ..., a_{8m}$ are the reals defining $\sigma_1, ..., \sigma_m$, then there are polynomials $p_1, ..., p_8$ in the a_i such that

$$w(\sigma_1, ..., \sigma_m) = \begin{pmatrix} p_1 + ip_2 & p_3 + ip_4 \\ p_5 + ip_6 & p_7 + ip_8 \end{pmatrix}.$$

Therefore $(\sigma_1, ..., \sigma_m) \in R_w^*$ if and only if $p_2 + p_8 = 0$. Since R_w^* is closed, if R_w^* failed to be nowhere dense then it would contain a nonempty open set. As in § 4, this implies that $p_2 + p_8$ is identically zero on $SL_2(\mathbf{C})^m$ or, equivalently, that the trace of $w(\sigma_1, ..., \sigma_m)$ is real for all $\sigma_1, ..., \sigma_m \in SL_2(\mathbf{C})$. This leads to a contradiction as follows.

A result of Magnus [19] and Neumann [30] (for a proof, see [20, § III.2]) states that the matrices $\rho = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $\tau = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ are free generators of a subgroup of $SL_2(\mathbf{Z})$ that consists only of the identity and hyperbolic elements. It follows that the same is true of the group generated by $\sigma_1, \sigma_2, \ldots$, where $\sigma_i = \rho^i \tau^i$. Now, for $z \in \mathbf{C} \backslash \{\frac{1}{2}(1 \pm \sqrt{5}), \frac{1}{4}(1 \pm \sqrt{5})\}$, define $\rho(z)$ and $\tau(z)$ in $SL_2(\mathbf{C})$ by

$$\rho(z) = \frac{1}{1 + z - z^2} \begin{pmatrix} 1 & z \\ z & 1+z \end{pmatrix} \quad \text{and} \quad \tau(z) = \frac{1}{1 + 2z - 4z^2} \begin{pmatrix} 1+2z & 2z \\ 2z & 1 \end{pmatrix}.$$

Let $\sigma_i(z) = \rho^i(z)\tau^i(z)$. Then choose a region Ω in $\mathbf{C}$ so that $0, 1 \in \Omega$ but Ω does not contain any of the 4 real singularities, and define a complex analytic function f on Ω by $f(z) = \operatorname{trace}(w(\sigma_1(z), ..., \sigma_m(z)))$. The assumption on w of the previous paragraph, together with the Open Mapping Theorem applied to f, yields that f is constant on Ω. But $f(0) = 2$ and $f(1)$ is the trace of a nonidentity element of the Magnus-Neumann group, whence $f(1) \neq 2$, a contradiction. Alternatively (as pointed out by a referee), one can obtain a contradiction by using Theorem 1 and Remark 4 of [5] to obtain that $f_w(SL_2(\mathbf{C})^m)$ has nonempty interior, whence the image of trace $\cdot f_w$ has nonempty interior in $\mathbf{C}$. Therefore the image of trace f_w is not contained in $\mathbf{R}$.

This lemma implies that R_w is nowhere dense too, so we may apply Theorem 2 to the collection $\{R_w\}$, yielding Theorem 1 (a) forH^3. This completes the proof of Theorem 1.

§ 7. Geometrical Consequences

In this section we summarize some striking geometrical consequences of the existence of large free groups. The following theorem illustrates what can be done with locally commutative actions. Unlike the preceding sections, the results of this section all use the Axiom of Choice. We use $D \Delta E$ to denote $(D \backslash E) \cup (E \backslash D)$.

THEOREM 4. *Suppose a free group, G, of rank $\kappa (\kappa \geqslant 2)$ is locally commutative in its action on X.*

(a) *If (and only if) $\kappa^\lambda = \kappa = |X|$, then there is a subset E of X such that for any $D \subseteq X$ with $|D| \leqslant \lambda$, there is some $\sigma \in G$ such that $\sigma(E) = E \Delta D$. In short, E is invariant under the addition and deletion of any λ points of X.*

(b) *X may be partitioned into κ sets, A_α, $\alpha < \kappa$, such that each A_α is G-equidecomposable with X using 2 pieces, i.e. for each α there are $\sigma_\alpha, \tau_\alpha \in G$ and $B_\alpha, C_\alpha \subseteq A_\alpha$ such that $\{B_\alpha, C_\alpha\}$ partitions A_α and $\{\sigma_\alpha(B_\alpha), \tau_\alpha(C_\alpha)\}$ partitions X. In short, X may be taken apart into pieces which may be rearranged to form κ copies of X.*

(c) *There is a subset E of X such that for any cardinal λ satisfying $3 \leqslant \lambda \leqslant \kappa$, X may be partitioned into λ G-congruent pieces, each of which is G-congruent to E. In short, E is, simultaneously, a third, a quarter, ..., a κ'th part of X.* (If the action is fixed-point free, then $\lambda = 2$ is also permitted — see Theorem 6.)

Parts (b) and (c) of this theorem are applications of a more general fact about locally commutative actions of a free group, which is described following Theorem 6.

Theorem 1 shows that all parts of the preceding theorem, with $\kappa = 2^{\aleph_0}$, apply to S^n, L^n and $H^n (n \geqslant 2)$ and $\mathbf{R}^n (n \geqslant 3)$, where G is either $G(X)$ or, in the case of L^n, the group of all isometries. Note that, since $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$, part (a) yields a set that is invariant under the addition or deletion of countably many points. Because the existence of large free locally commutative groups was already known in most of these cases, so were the consequences by Theorem 4; only the cases of S^4 and L^4 are new.

Part (a) is due to Mycielski [24]. It is known to be false in $\mathbf{R}^1$, $\mathbf{R}^2$ and S^1 even if one only seeks invariance with respect to the deletion of single points (Sierpiński [37], Straus [38]). Under appropriate (and necessary) assumptions about cardinal arithmetic, part (a) can be used to get sets invariant under the addition and deletion of certain uncountable sets of points. For example, the (consistent) assumption that $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ implies that $(2^{\aleph_0})^{\aleph_1} = 2^{\aleph_0}$, so part (a) is valid with $\kappa = 2^{\aleph_0}$ and $\lambda = \aleph_1$. The proof of Theorem 4 (a) uses the Axiom of Choice, but it is not known whether the set E must necessarily be nonmeasurable.

Part (b) is a refinement of the classical Banach-Tarski Paradox on S^2 along lines first investigated by Robinson [34] and Sierpiński [36]. As stated above, the result is due to Dekker [7], who also proved the following converse.

THEOREM 5. *Suppose* $\kappa \geqslant 2$ *and the action of* G *on* X *satisfies assertion (b) of Theorem 4. Then* G *contains a free subgroup of rank* κ *whose action on* X *is locally commutative; indeed* $\sigma_\alpha^{-1}\tau_\alpha$, $\alpha < \kappa$, *freely generate such a subgroup.*

Work of Banach and von Neumann (see [27]) yields that a solvable group is amenable and whenever an amenable group G acts on X then there exists a finitely additive G-invariant measure μ defined on all subsets of X, with $\mu(X) = 1$. This implies that Theorem 4 (b) is not valid for S^1, $\mathbf{R}^1$ or $\mathbf{R}^2$, even for $\kappa = 2$.

Part (c) of Theorem 4 (Mycielski [22]) is a generalization of an earlier result of Robinson [34], who showed that S^2 may be divided into 3 (or n, if $3 \leqslant n < \aleph_0$) rotationally congruent pieces. It is not clear that Robinson's result requires nonmeasurable pieces, and the following problem (Mycielski [23]) is still unsolved.

Problem. Can S^2 be partitioned into 3 rotationally congruent, Lebesgue measurable sets?

The assertion of 4 (c), however, does necessitate nonmeasurable pieces in S^n and $\mathbf{R}^n$ (for the latter, and for the case of H^n, see § 8). Hence, for the same reasons as for 4 (b), 4 (c) is false in S^1, $\mathbf{R}^1$ and $\mathbf{R}^2$. However, for any $\lambda \leqslant 2^{\aleph_0}$, S^1 may be partitioned into λ pairwise congruent pieces (see [40]). Note that $\lambda = 2$ is omitted from part (c); this is because every element of SO_3, for example, has a fixed point in S^2, therefore S^2 cannot be split into two SO_3-congruent pieces.

Parts (b) and (c) of Theorem 4 are related to the solution of certain systems of congruences. The following theorem (Dekker [7]) shows that a fixed-point free action allows a wide variety of such systems to be solved.

THEOREM 6. *Suppose the action of* G, *a free group of rank* κ, *on* X *is fixed-point free and* $\{\cup\{A_\alpha : \alpha \in L_\beta\} \equiv \cup\{A_\alpha : \alpha \in R_\beta\} : \beta < \kappa\}$ *is a system of* κ *congruences, where each* L_β *and* R_β *is a proper and nonempty subset of* λ. *Then* X *can be partitioned into sets* A_α, $\alpha < \lambda$, *so that each congruence in the system is witnessed by some free generator of* G.

A similar result is true for locally commutative actions, but one has to restrict the systems of congruences to those systems which do not, explicitly or implicity, imply that a set is congruent to its complement. Parts (b) and (c) of Theorem 4 are consequences of this general result. For example, to obtain (b) consider the system.

$$\{A_\alpha \equiv \cup\{A_\beta : \beta < \kappa,\ \beta \neq \alpha + 1\} : \alpha < \kappa,\ \alpha \text{ even}\}$$

and, for $\alpha < \kappa$, α even, let $B_\alpha = A_\alpha$, $C_\alpha = A_{\alpha+1}$.

Because of Theorem 1, Theorem 6, with $\kappa = 2^{\aleph_0}$, applies to S^n and L^n if $n \geqslant 3$ and n is odd, and to H^n and $\mathbf{R}^n$ if $n \geqslant 3$. Moreover, it applies to H^2 if $\kappa = \aleph_0$. Since, as just shown, the conclusion of Theorem 6 implies the assertion of Theorem 4 (b), it follows from Theorem 5 that a partial converse to Theorem 6 is valid: if an action admits a solution to all κ-sized systems of congruences, then G contains a free locally commutative subgroup of rank κ. But the stronger converse to Theorem 6, i.e., the existence of a fixed-point free subgroup, is false. This follows from work of Adams [1], who showed that if the antipodal map from S^n to S^n is available, as it is in SO_{2n} or any O_n, then a locally commutative free group is sufficient to obtain the conclusion of Theorem 6, provided no element of the locally commutative group has -1 as an eigenvalue. This latter condition is clearly satisfied by a free subgroup of SO_3, so Adams' theorem yields the conclusion of Theorem 6 for the action of O_3 on S^2, with $\kappa = 2^{\aleph_0}$. But no free subgroup of O_3 is fixed-point free in its action on S^2.

Because no elements of the locally commutative free subgroups of SO_n constructed by Dekker [7] and Borel [5] have -1 as an eigenvalue, Adams' technique yields the conclusion of Theorem 6, with $\kappa = 2^{\aleph_0}$, for the action of O_{n+1} on S^n, for all $n \geqslant 2$. In fact, any non-Abelian locally commutative free subgroup of SO_3, SO_4 or SO_5 must avoid -1 as an eigenvalue. For SO_3 this is clear since a rotation that sends a point to its

antipode must have order 2. Suppose $\sigma, \tau \in SO_5$ freely generate a locally commutative group and some word w has -1 as an eigenvalue. Then this eigenvalue must have multiplicity 2, whence w^2 fixes a 3-dimensional subspace of $\mathbf{R}^5$. Assume w^2 does not begin, on the left, with $\sigma^{\pm 1}$ and let $u = \sigma w^2 \sigma^{-1}$. By freeness, u and w^2 are not powers of a common word; therefore u and w^2 do not commute (see [21, p. 42]). But u also fixes a 3-dimensional subspace, so u and w^2 must share a fixed point on the unit sphere, which contradicts local commutativity. A similar argument works in $\mathbf{R}^4$: choose a basis consisting of two linearly independent fixed points of w^2 and two linearly independent fixed points of u; it follows that u and w^2 commute. These arguments lead to the following question.

Problem. Does SO_6 (or SO_n, $n \geqslant 6$) have a locally commutative free subgroup of rank 2 which contains a transformation having -1 as an eigenvalue?

As an application of Theorem 6, consider the result of Theorem 4 (c). A solution of the following system of $2^{\aleph_0}$ congruences involving A_α, $\alpha < 2^{\aleph_0}$, yields a set E satisfying Theorem 4 (c) for any λ such that $2 \leqslant \lambda \leqslant 2^{\aleph_0}$:

$$A_0 \equiv A_\beta, \quad \beta < 2^{\aleph_0}$$

$$A_\beta \equiv \cup\{A_\alpha : \beta < \alpha < 2^{\aleph_0}\}, \quad \beta < 2^{\aleph_0}.$$

Hence, using Adams' result (when necessary), we obtain the following corollary to Theorems 1 and 5.

COROLLARY. *Let X be any of S^n, $n \geqslant 3$, n odd, or $\mathbf{R}^n$ or H^n, with $n \geqslant 3$, and let $G = G(X)$. Or, let X be S^n, $n \geqslant 2$ or L^n, $n \geqslant 3$, n odd, with G being the group of all isometries of X. Then there is a subset E of X such that, for any λ with $2 \leqslant \lambda \leqslant 2^{\aleph_0}$, X may be split into λ sets, each of which is G-congruent to E.*

Because of the anomaly about H^2 discussed in § 6, it is not known whether the conclusion of Theorem 6 is valid in H^2 for some uncountable κ. In particular, we have the following problem, where a set is called a λ'th part of H^2 if H^2 splits into λ sets, each of which is congruent, via $PSL_2(\mathbf{R})$, to the set.

Problem. Does H^2 contain a set which is both a half of H^2 and a $2^{\aleph_0}$'th part of H^2?

Note, however, that because Theorem 6 is valid in H^2 with $\kappa = \aleph_0$ there is a subset of H^2 (indeed, a Borel set; see § 8) that is both a half of H^2 and an $\aleph_0$'th part of H^2; consider the set of congruences preceding the corollary based on the set-variables $\{A_n : n < \aleph_0\}$. Moreover, The-

orems 1 (c) and 4 (c) yield a subset that is both a third of H^2 and a $2^{\aleph_0}$'th part of H^2.

§ 8. A Paradoxical Decomposition Using Borel Sets

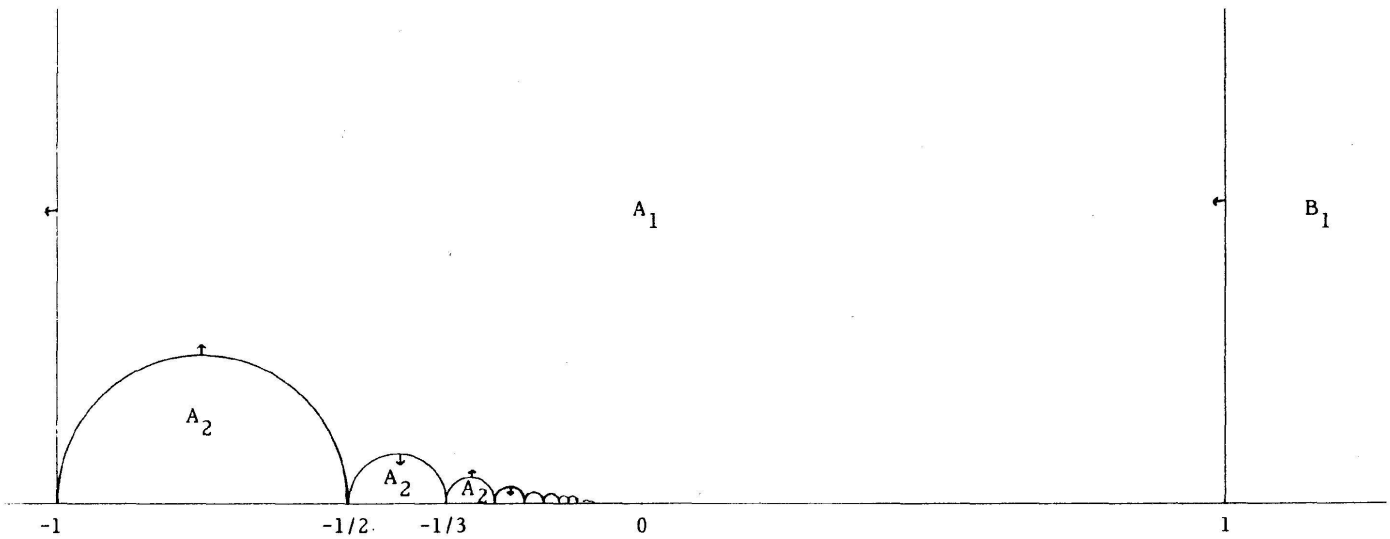
THEOREM 8. *If $n \geqslant 2$, then any system of countably many congruences involving countably many sets (as in Theorem 6) is satisfiable using a partition of H^n into Borel sets and isometries.*

Proof. Consider H^2 first, and let F be a free subgroup of $PSL_2(\mathbf{Z})$ whose rank equals the number of congruences to be satisfied; F may be obtained as a subgroup of the group generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and its transpose. Theorem 6 is proved by first constructing, by induction, a partition of F that satisfies the given system using left multiplication in F. Then it is easy to transfer this decomposition to a set on which F's action is fixed-point free by using a choice set for the F-orbits. In general, this requires the Axiom of Choice, and yields nonmeasurable sets. But, because F is a discrete subgroup of $PSL_2(\mathbf{R})$, there is a fundamental region for F's action on H^2. In fact (see [18]) there is a (hyperbolic) polygon such that no two points of the polygon's interior lie in the same F-orbit, and all points in H^2 are in the F-orbit of some point in the closure of the polygon. The boundary of this polygon consists of a countable number of sides (open hyperbolic segments) and vertices. Since F maps vertices to vertices and sides to sides, there is a choice set M for the F-orbits that consists of the interior of the polygon together with some of the vertices and some of the sides. Clearly, M is a Borel set. Now, if B_n is one of the sets of the partition of F, then let $A_n = \cup\{\sigma(M) : \sigma \in B_n\}$. This yields a partition of H^2 into Borel sets A_n which satisfy the given congruences. The result for higher dimensions follows by simple using the standard projection of H^n onto H^2 to define the pieces of a partition of H^n.

COROLLARY. *If $n \geqslant 2$ then H^n is paradoxical using Borel sets. In fact, there are pairwise disjoint Borel sets, A_1, A_2, B_1, B_2 and isometries $\sigma_1, \sigma_2, \tau_1, \tau_2 \in G(H^n)$ such that $H^n = \sigma_1(A_1) \cup \sigma_2(A_2) = \tau_1(B_1) \cup \tau_2(B_2)$. Moreover, there is a Borel set E which is simultaneously a half, a third, ..., an $\aleph_0$'th part of H^2.*

This corollary shows that the subsets of H^n provided by parts (b) of (c) of Theorem 4 can be taken to be Borel sets in the case $\kappa = \aleph_0$. This

result is completely constructive. For instance, if one labels the quadrilaterals of the tesselation corresponding to the discrete free group generated by σ and τ (where $\sigma(z) = \dfrac{z}{2z+1}$ and $\tau(z) = z + 2$) and then transfers the paradoxical decomposition of a free group of rank two to H^2 via the labelled quadrilaterals, one obtains the partition of H^2 into four sets A_1, A_2, B_1 and B_2 illustrated in the figure below. Since $H^2 = A_1 \cup \sigma(A_2) = B_1 \cup \tau(B_2)$, this yields an explicit paradoxical decomposition of the hyperbolic plane using very simple sets. For another pictorially simple paradox in H^2 see [41, Fig. 5.2].



These results are completely opposite to the situation in S^2 and $\mathbf{R}^n$. Because of surface Lebesgue measure on S^n, it is obvious that parts (b) and (c) of Theorem 4 cannot be witnessed by measurable sets. For example, if m denotes surface Lebesgue measure and E, a measurable set, is a λ'th part of S^n, then $m(E) = \dfrac{1}{\lambda}$, if λ is finite, and $m(E) = 0$ if λ is infinite. The case of $\mathbf{R}^n$ is subtler because $\mathbf{R}^n$ has infinite measure; the following result of Mycielski [27] is relevant.

THEOREM 9. *There is a finitely additive measure* μ *on the collection of Lebesgue measurable subsets of* $\mathbf{R}^n$ *which is invariant under all similarities and satisfies* $\mu(\mathbf{R}^n) = 1$.

Because the similarity groups in $\mathbf{R}^1$ and $\mathbf{R}^2$ are solvable, the theorem of Banach mentioned in § 7 shows that, in these two cases, the measure can be taken to be defined on all sets.

Note that for κ uncountable parts (b) and (c) of Theorem 4 cannot be witnessed by Borel subsets of H^n. Suppose, for example, that κ is uncountable

and the sets of Theorem 4 (b) are all Borel. Since Borel sets have the Property of Baire, each A_α may be written as $R_\alpha \Delta M_\alpha$ where R_α is open and M_α is meager. But each A_α, being Borel equidecomposable to all of H^2, is nonmeager, whence each R_α is nonempty. It follows that the R_α are pairwise disjoint, which contradicts the separability of H^2. A similar argument shows that the sets cannot all be Lebesgue measurable either.

Let us point out how the proof of Theorem 9 breaks down in hyperbolic space. Theorem 9 is based on the fact that $\mathbf{R}^n$ is a union of countably many sets B_r of finite Lebesgue measure satisfying: for any isometry σ, $m(B_r \Delta \sigma(B_r))/m(B_r) \to 0$ as $r \to \infty$. Simply let B_r be the ball of radius r centered at the origin. Because Theorem 9 is false for H^n if $n \geqslant 2$, there can be no such sequence of almost invariant sets of finite (hyperbolic) measure in H^n.

§ 9. Linear Transformations of the Euclidean Plane

Paradoxical decompositions in the plane are possible if one allows the use of area-preserving affine transformations. This was first realized by von Neumann [31], who showed that a square is paradoxical using this expansion of the isometry group. In fact, it is sufficient to consider the group generated by $SL_2(\mathbf{Z})$ and all translations; see [39]. In this section we discuss how the results of this paper are affected by considering linear, or affine, transformations instead of just isometries.

Let us consider the action of $SL_2(\mathbf{R})$ on $\mathbf{R}^2\backslash\{0\}$. The two matrices, $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$ freely generate a subgroup of $SL_2(\mathbf{Z})$, no nonidentity element of which has a fixed point in $\mathbf{R}^2\backslash\{0\}$; this follows from the result of Magnus and Neumann mentioned in § 6, since an element of $SL_2(\mathbf{Z})$ has a nonzero fixed point in $\mathbf{R}^2$ if and only if it has trace 2. It follows by the technique of § 4 that $SL_2(\mathbf{R})$ has a free subgroup with a perfect set of free generators whose action on $\mathbf{R}^2\backslash\{0\}$ is fixed-point free. Therefore the action of $SL_2(\mathbf{R})$ on $\mathbf{R}^2\backslash\{0\}$ satisfies all the conclusions of Theorems 4 and 6.

Using techniques of functional analysis, J. Rosenblatt and R. Kallman (unpublished) have recently shown that the Lebesgue measurable subsets of $\mathbf{R}^n\backslash\{0\}$ ($n \geqslant 2$) do not bear a finitely additive, $SL_n(\mathbf{Z})$-invariant measure of total measure one. (For $n \geqslant 3$ this uses the fact that $SL_n(\mathbf{Z})$ has Kazhdan's Property T, while the $\mathbf{R}^2$ case uses specific facts about representations of

$SL_2(\mathbf{Z})$; see [41; Theorem 11.17].) Thus Theorem 9 does not extend to area-preserving affine transformations. It would be interesting if a paradoxical decomposition of $\mathbf{R}^2\backslash\{0\}$ using measurable sets, similar to the one illustrated in § 8, could be explicitly constructed. Some sort of paradoxical decomposition using measurable pieces must exist, by a general theorem of Tarski (see [41]), but it is not known if one using just four pieces exists. On the other hand, Belley and Prasad [4] have shown that there is a finitely additive measure on a certain (not too small) Boolean algebra of Borel subsets of $\mathbf{R}^n$ that has total measure one and is invariant under all nonsingular affine transformations of $\mathbf{R}^n$ (not just the measure-preserving ones).

Finally, we mention some unsolved problems about the existence of nice free groups of affine, area-preserving transformations, positive solutions to which would yield (via Theorems 4-6) paradoxical decompositions of $\mathbf{R}^n$. Let $A_n(\mathbf{R})$ denote the group of affine transformations of $\mathbf{R}^n$, i.e., transformations of the form TL, where T is a translation and $L \in GL_n(\mathbf{R})$. Let $SA_n(\mathbf{R})$ be the subgroup obtained by restricting L to $SL_n(\mathbf{R})$, and let $SA_n(\mathbf{Z})$ consist of those TL where $L \in SL_n(\mathbf{Z})$ and T is a translation by a vector in $\mathbf{Z}^n$. Note that $SA_n(\mathbf{Z})$ acts on $\mathbf{Z}^n$. Since $G(\mathbf{R}^3) \subseteq SA_3(\mathbf{R})$, Theorem 1 yields that $SA_3(\mathbf{R})$ has a free non-Abelian subgroup whose action on $\mathbf{R}^3$ is fixed-point free. Consideration of $\mathbf{Z}^3$ instead of $\mathbf{R}^3$ leads to problem 1 below. Problem 2 is an attempt to get a version of these results for $\mathbf{R}^2$ (rather than $\mathbf{R}^2\backslash\{0\}$, which is treated at the beginning of this section). Only local commutativity is sought because of part (b) of the proposition below. Since $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and its transpose freely generate a group of rank two, so do the two transformations:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence perhaps the subgroup of $SA_2(\mathbf{Z})$ which these two transformations generate solves Problem 2 affirmatively. But we are unable to show that this subgroup is locally commutative.

Problems.

1. Does $SA_3(\mathbf{Z})$ have a free subgroup of rank two which is fixed-point free on $\mathbf{Z}^3$?

2. Does $SA_2(\mathbf{R})$ (or $SA_2(\mathbf{Z})$) have a subgroup of rank two which is locally commutative in its action on $\mathbf{R}^2$ (or on $\mathbf{Z}^2$)?

PROPOSITION 10.

(a) *If* $TL \in A_n(\mathbf{R})$ *and* TL *has no fixed points in* $\mathbf{R}^n$, *then* L *has* $+1$ *as an eigenvalue, i.e.,* L *has a fixed point in* $\mathbf{R}^n \backslash \{0\}$.

(b) *If* G *is a subgroup of* $SA_2(\mathbf{R})$ *which is fixed-point free on* $\mathbf{R}^2$ *then* G *is solvable.*

Proof.

(a) Suppose T is a translation by the vector v. Since $L(x) + v = x$ has no solution, the same is true of $(L-I)(x) = -v$, and therefore $\det(L-I) = 0$, i.e., 1 is an eigenvalue of L.

(b) Let $\sigma = TL$ and $\tau = T'L'$ be in G. Then $\sigma\tau = T''LL'$ so part (a) yields that each of L, L', LL' has 1 as an eigenvalue. Since these are 2×2 matrices with determinant 1, this implies that all have trace 2. Hence, choosing an appropriate basis, we have $L = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and $L' = \begin{pmatrix} \alpha & \beta \\ \gamma & 2-\alpha \end{pmatrix}$. Then $LL' = \begin{pmatrix} \alpha + b\gamma & * \\ * & 2-\alpha \end{pmatrix}$, and the trace of the latter being 2 yields that $b\gamma = 0$. But if either b or γ equal zero, then L and L' commute, which implies that the commutators $\sigma\tau\sigma^{-1}\tau^{-1}$ and $\sigma^{-1}\tau^{-1}\sigma\tau$ are pure translations. Hence [[G, G], [G, G]] is the identity subgroup, i.e., G is solvable.

Part (b) of the Proposition shows why there is no fixed-point free, non-Abelian free subgroup of $SA_2(\mathbf{R})$. But the following problem is unsolved.

Problem 3. Does there exist a free non-Abelian semigroup in $SA_2(\mathbf{R})$ (or $SA_2(\mathbf{Z})$) whose action on $\mathbf{R}^2$ is fixed-point free?

Part (a) of Proposition 10 brings to light a distinction between the groups $G(\mathbf{R}^n)$ according as n is even or odd. The proof of Theorem 1 for $\mathbf{R}^3$ (§ 5) is essentially the same as the proof for S^{2n+1} given in § 4. Precisely, it is shown that $A = \{\sigma \in G(\mathbf{R}^3) : \sigma \text{ has a fixed point in } \mathbf{R}^3\}$ is nowhere dense and, in fact, each $R_w = f_w^{-1}(A)$ is nowhere dense in the appropriate product, where w is any group word in finitely many variables. While this is sufficient to get the existence of perfect free generating sets of fixed-point free subgroups in $\mathbf{R}^3$ and beyond, the set A can fail to be nowhere dense in the higher dimensions. Indeed, consider $\mathbf{R}^{2n}$, $n \geqslant 1$. Letting $\pi: G(\mathbf{R}^{2n}) \to SO_{2n}$ be the canonical homomorphism, it follows from part (a) of Proposition 10 that $G(\mathbf{R}^n) \backslash A \subseteq \pi^{-1}(B)$, where $B = \{L \in SO_{2n} : L \text{ has 1 as an eigenvalue}\}$. It is easy to see that B is nowhere dense and it follows that the same is true of $\pi^{-1}(B)$; i.e., A has a nowhere dense complement. In odd

dimensions, however, the situation in $\mathbf{R}^3$ is typical, as the following proposition shows.

PROPOSITION 11. *If* $n \geqslant 1$ *is odd then* $A = \{\sigma \in G(\mathbf{R}^n) : \sigma$ *has a fixed point in* $\mathbf{R}^n\}$ *is a nowhere dense subset of* $G(\mathbf{R}^n)$.

Proof. It is an easy linear algebra exercise (generalizing Proposition 10 (a) above) to see that $\sigma = TL$ has a fixed point in $\mathbf{R}^n$ if and only if the translation vector of T is orthogonal to all vectors fixed by L. Since there is a basis for the fixed space of L that consists of vectors whose entries are polynomials in the entries of L (Gaussian elimination and scaling), this latter condition on TL is equivalent to the vanishing of a polynomial in the entries of σ. But the condition is not universally true in $G(\mathbf{R}^n)$ since any pure translation has no fixed points; therefore the technique introduced in § 4 implies that A is nowhere dense, as desired.

This proposition, in exactly the same cases, is valid for SO_{n+1}'s action on S^n (see § 4). The following extension of these results is a refinement of the theorems on the existence of free, fixed-point free groups of isometries of rank m: it shows that in these cases almost all (from the category point of view) m-tuples of isometries are free generators of fixed-point free groups of isometries.

PROPOSITION 12. *Suppose* n *is odd and* $n \geqslant 3$, *and* X *is one of* $\mathbf{R}^n$ *or* S^n. *Then any* m *elements of* $G(X)$, *with the exception of a meager set in* $G(X)^m$, *are free generators of a fixed-point free subgroup of* $G(X)$.

Proof. For the spherical case this follows from § 4, where it was shown that $\cup\{R_w : w$ is a group word in m variables$\}$ is comeager. The Euclidean case is proved by observing (see Proposition 11's proof and § 5) that there is a function p that is a polynomial in the entries of $\sigma_1, \ldots, \sigma_m$ such that $p = 0$ if and only if $f_w(\sigma_1, \ldots, \sigma_m) \in A$. Since, by the rank two case of Theorem 1 (a), f is not identically zero, $f_w^{-1}(A)$ is nowhere dense. Therefore the union over all words in m variables is meager, as desired.

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