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Autor: Farkas, Daniel R.
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§ 4. Appendix

$R = R_{(0)} \oplus R_{(1)} \oplus \ldots$ is a graded k-algebra with $R_{(0)} = k$. Let $\mathfrak{m}$ be the maximal ideal $\sum_{i=1}^{\infty} R_{(i)}$. We assume that $\hat{R}$ is a power series ring in finitely many variables. Obviously $\hat{\mathfrak{m}}$ corresponds to the unique maximal ideal of the power series ring, whence $\hat{R}/\hat{\mathfrak{m}}^d$ is always finite dimensional. Since $\hat{\mathfrak{m}}^d$ is homogeneous, some tail $\prod_{?}^{\infty} R_{(i)}$ must then lie in $\hat{\mathfrak{m}}^d$. It follows that the graded algebra of R for the $\mathfrak{m}$-adic filtration is isomorphic to the graded algebra of $\hat{R}$ for the $\hat{\mathfrak{m}}$-adic filtration. The power series assumption implies that the latter is simply a polynomial ring with the standard grading.

Clearly $\mathfrak{m}^2 \subset \sum_{j=2}^{\infty} R_{(j)}$. Hence $R_{(1)}$ injects into $\mathfrak{m}/\mathfrak{m}^2$. Choose a basis for $R_{(1)}$ over k and extend it to a list of homogeneous elements $x_1, \ldots, x_n$ in $\mathfrak{m}$ whose images constitute a basis for $\mathfrak{m}/\mathfrak{m}^2$. It is generally true for any commutative k-algebra R that when $R/\mathfrak{m} = k$ and when the associated graded ring for the $\mathfrak{m}$-adic filtration is the symmetric algebra on $\mathfrak{m}/\mathfrak{m}^2$, that any basis for $\mathfrak{m}/\mathfrak{m}^2$ pulls back to a set of algebraically independent elements in R. In particular, $x_1, \ldots, x_n$ are algebraically independent.

We use the given grading on R to prove that $R = k[x_1, \ldots, x_n]$. Vacuously, $R_{(0)} \subset k[x_1, \ldots, x_n]$. We have chosen the x_i so that $R_{(1)}$ lies in their span, so $R_{(1)} \subset k[x_1, \ldots, x_n]$. Assume, inductively, that $d \geqslant 1$ and $R_{(s)} \subset k[x_1, \ldots, x_n]$ for all $s \leqslant d$. If $y \in R_{(d+1)}$ then

$$y = \Sigma \lambda_i x_i + \Sigma u_j v_j$$

for some $\lambda_i \in k$ and u_j, $v_j \in \mathfrak{m}$. Without loss of generality u_j and v_j are homogeneous and all the x_i and $u_j v_j$ which appear in the formula lie in $\bigcup_{t=1}^{d+1} R_{(t)}$. This can only happen when u_j and v_j are in $R_{(s)}$ for some $s \leqslant d$. By induction, u_j and v_j are elements of $k[x_1, \ldots, x_n]$. Therefore $y \in k[x_1, \ldots, x_n]$.

§ 5. Weyl Groups

It seems to be-part of the folklore for Lie theory that the converse of Theorem 8 fails to be true (cf. [4] VI§ 3 Ex. 2). Rather than being dead-ends, these examples serve as inspiration: the machinery of root systems will allow us to determine the correct necessary and sufficient conditions for a multiplicative Shephard-Todd-Chevalley analogue. For the most part, we will follow the notation in [8].

Suppose that V is an n-dimensional complex vector space and $G \subset GL(V)$. By a G-lattice we mean a lattice in V (of rank n) which is invariant under the action of G. The G-lattice A is effective if zero is the only element fixed by all members of G. Notice that A is effective if and only if the units of $\mathbf{C}[A]^G$ are precisely the nonzero elements of $\mathbf{C}$.

PROPOSITION 12. *Let A be an effective G-lattice. If G is a finite group generated by reflections then*

(i) *there is a reduced root system Φ lying in A so that G is the Weyl group for Φ, and*

(ii) *A (considered inside V) lies between the root lattice for Φ and the weight lattice.*

Proof. Endow V with an inner product which makes members of G orthogonal transformations. If σ is a reflection in G and $a \in A$ is such that $a \neq \sigma(a)$ then $a - \sigma(a) \neq 0$ and $\sigma(a - \sigma(a)) = -(a - \sigma(a))$. Thus $\{b \in A \mid \sigma(b) = -b\}$ is an infinite cyclic subgroup of A. Its two possible generators, a_σ and $-a_\sigma$, are the nonzero vectors of smallest length in A which are "reflected" by σ. It is not difficult to check that $\Phi = \{\pm a_\sigma \mid \sigma \text{ is a reflection in } G\}$ is a root system, whence G is its Weyl group. Moreover, if $x \in A$ and $\alpha = \pm a_\sigma \in \Phi$ then $\sigma(x) \in A$. Thus $x - \dfrac{2(x, \alpha)}{(\alpha, \alpha)} \alpha \in A$. Now $\dfrac{2(x, \alpha)}{(\alpha, \alpha)} \alpha \in A$ implies that $\dfrac{2(x, \alpha)}{(\alpha, \alpha)}$ is an integer. This is just the statement that x is a weight. □

Although we have "located" the effective G-lattices, there are still quite a few of them: every lattice between the root lattice and the weight lattice is invariant under G. On the positive side, it turns out that the group algebra of the weight lattice has well-behaved invariants.

THEOREM ([4], VI § 3.4). *Let G be a Weyl group and let Λ be its weight lattice. Then $\mathbf{C}[\Lambda]^G$ is a polynomial ring.* □

This theorem of Bourbaki can be generalized just enough to suggest its own converse. Fix a root system with base Δ. Let Λ_r and Λ denote the root lattice and weight respectively and let $w_1, \ldots, w_n$ be the fundamental dominant weights. Then Λ^+ is the collection of dominant weights: the non-negative integer combinations of $w_1, \ldots, w_n$. Write W for the Weyl group.

In [5], we introduced the notion of *stretched weight lattice* for a root system. It is a W-lattice lying between Λ_r and Λ which has a basis of the form $r_1 w_1, r_2 w_2, \ldots, r_n w_n$ for positive integers $r_1, \ldots, r_n$. A stretched weight lattice can always be built up from ordinary weight lattices and certain root lattices ([5]). More unexpectedly, we found an abstract characterization. Suppose G is a finite subgroup of $GL(n, \mathbf{Z})$; then the corresponding action on $\mathbf{Z}^n$ has the non-negative "quadrant" as fundamental domain (in Bourbaki's strong sense) if and only if G is a Weyl group and $\mathbf{Z}^n$ is isomorphic to a stretched weight lattice for G.

To talk about the group algebra $\mathbf{C}[\Lambda]$, we will have to switch from additive to multiplicative notation for elements of Λ. If we think of λ as a weight then λ^* will be its image in $\mathbf{C}[\Lambda]$, e.g. $(\lambda_1 - \lambda_2)^* = (\lambda_1^*)(\lambda_2^*)^{-1}$.

For $\lambda \in \Lambda$ we set $X(\lambda) = (\text{constant}) \cdot \mathrm{av}(\lambda^*)$ where the normalizing constant is chosen so that each element of Λ appears with coefficient 0 or 1 in $X(\lambda)$. Using this notation, we state the appropriate form of Bourbaki's Theorem. (The proof carries over verbatim from [4].)

THEOREM 13. *If S is a stretched weight lattice with basis $r_1 w_1, \ldots, r_n w_n$ then*

$$\mathbf{C}[S]^W = \mathbf{C}[X(r_1 w_1), \ldots, X(r_n w_n)].$$

Moreover, $X(r_1 w_1), \ldots, X(r_n w_n)$ are algebraically independent. □

We shall frequently use the consequence that $X(w_1), \ldots, X(w_n)$ are irreducible elements of the unique factorization domain $\mathbf{C}[\Lambda]^W$.

For the rest of this paper, M will be a W-lattice with

$$\Lambda_r \subset M \subset \Lambda .$$

LEMMA 14. *Suppose $\lambda_1, \ldots, \lambda_t$ are (not necessarily distinct) dominant weights. If $\lambda_1 + \ldots + \lambda_t \in M$ then $(g_1 \cdot \lambda_1) + \ldots + (g_t \cdot \lambda_t) \in M$ for all choices $g_1, \ldots, g_t \in W$.*

Proof. For $\alpha \in \Delta$ let σ_α denote reflection in the hyperplane perpendicular to α. Then $\sigma_\alpha(\lambda_j) = \lambda_j - < \lambda_j, \alpha > \alpha$. The definition of "weight" implies that $< \lambda_j, \alpha >$ is an integer. Thus

$$\sigma_\alpha(\lambda_j) \equiv \lambda_j \pmod{\Lambda_r}$$

and so,

$$\sigma_\alpha(\lambda_j) \equiv \lambda_j \,(\mathrm{mod}\ M)\,.$$

Now W is generated by $\{\sigma_\alpha \mid \alpha \in \Delta\}$. An easy induction on the length of $g \in W$ as a word in the generators yields

$$g(\lambda_j) \equiv \lambda_j \,(\mathrm{mod}\ M)\,.$$

Hence

$$\sum_{j=1}^{t} g_j(\lambda_j) \equiv \sum_{j=1}^{t} \lambda_j \,(\mathrm{mod}\ M)\,. \qquad \square$$

LEMMA 15. *Suppose* $\lambda_1, \ldots, \lambda_t$ *are (not necessarily distinct) dominant weights. If* $\lambda_1 + \ldots + \lambda_t \in M$ *then*

$$X(\lambda_1)X(\lambda_2) \cdots X(\lambda_t) \in \mathbf{C}[M]^W\,.$$

Proof. A typical element of Λ in the support of $X(\lambda_1) \cdots X(\lambda_t)$ has the form $(g_1(\lambda_1)+\ldots+g_t(\lambda_t))^*$ where $g_1, \ldots, g_t \in W$. According to Lemma 14, $\Sigma g_j(\lambda_j) \in M$. Thus

$$X(\lambda_1)X(\lambda_2) \cdots X(\lambda_t) \in \mathbf{C}[M] \cap \mathbf{C}[\Lambda]^W\,. \qquad \square$$

We say that an element $w \in M \cap \Lambda^+$ is *M-indecomposable* if it cannot be written as a sum of two nonzero elements of $M \cap \Lambda^+$. Clearly, every element of $M \cap \Lambda^+$ is a sum of M-indecomposable elements.

THEOREM 16. *The following statements are equivalent:*

(i) M *is a stretched weight lattice for* W.

(ii) $\mathbf{C}[M]^W$ *is a polynomial ring.*

(iii) $\mathbf{C}[M]^W$ *is a UFD.*

Proof. (i) $\Rightarrow$ (ii) is Theorem 13 and (ii) $\Rightarrow$ (iii) is classical. Thus we assume that $\mathbf{C}[M]^W$ is a *UFD* and prove (i).

Suppose $\sum_{j=1}^{n} a_j w_j$ is M-indecomposable. According to Lemma 15,

$$Y = X(w_1)^{a_1} X(w_2)^{a_2} \cdots X(w_n)^{a_n}$$

is an element of $\mathbf{C}[M]^W$. Every coefficient appearing in $X(w_j)$ is 1; hence any subproduct

$$X(w_1)^{b_1} X(w_2)^{b_2} \cdots X(w_n)^{b_n}$$

with $0 \leqslant b_j \leqslant a_j$ contains $(\sum_{j=1}^{n} b_j w_j)^*$ in its support. If Y factors in $\mathbf{C}[M]^W$ then each factor is one such subproduct by the UFD property of $\mathbf{C}[\Lambda]^W$. Therefore, a factoring provides b_j for $j = 1, \ldots, n$ such that $0 \leqslant b_j \leqslant a_j$, not all $b_j = a_j$, and both $\Sigma b_j w_j$ and $\Sigma(a_j - b_j)w_j$ lie in M. This contradicts the M-indecomposability of $\Sigma a_j w_j$. In summary, Y is an irreducible element in $\mathbf{C}[M]^W$.

Let d be the index of M in Λ. Then $dw_j \in M$ for each fundamental dominant weight w_j. Again, Lemma 15 yields

$$X(w_j)^d \in \mathbf{C}[M]^W \quad \text{for} \quad j = 1, \ldots, n\,.$$

Consider the equation

$$Y^d = [X(w_1)^d]^{a_1}[X(w_2)^d]^{a_2} \cdots [X(w_n)^d]^{a_n}$$

inside $\mathbf{C}[M]^W$. Since Y is irreducible, $Y \mid X(w_k)^d$ for some k. Interpret this in $\mathbf{C}[\Lambda]^W$ and use unique factorization there: $Y = X(w_k)^{a_k}$. That is, the M-indecomposable weights all have the form $a_k w_k$.

If $a_k w_k$ and $a'_k w_k$ lie in M, so does $GCD(a_k, a'_k)w_k$. But $GCD(a_k, a'_k)$ divides both a_k and a'_k. By indecomposability, there are no such repeats:

$$r_1 w_1, \ldots, r_n w_n \qquad (r_j > 0 \text{ an integer})$$

is a complete list of the M-indecomposable elements. (Notice that some positive integer multiple of each w_j *must* be M-indecomposable.) They are clearly linearly independent over $\mathbf{Z}$. The argument is completed by showing that they span M. Suppose $\sum_{i=1}^{n} c_i w_i \in M$. Choose a large positive integer N such that $\frac{c_i}{r_i} \leqslant N$ for $i = 1, \ldots, n$. Since $r_i w_i \in M$ we have $N(\sum_{i=1}^{n} r_i w_i) \in M$. Thus $\sum_{i=1}^{n} (Nr_i - c_i)w_i \in M$. Since $Nr_i - c_i \geqslant 0$,

$$\sum_{i=1}^{n} (Nr_i - c_i)w_i \in M \cap \Lambda^+\,.$$

Now every member of $M \cap \Lambda^*$ is a sum of M-indecomposable elements. Solve for $\Sigma c_i w_i$. □

Finally, we can put together Theorem 8, Proposition 12, and Theorem 16. We cite the fact that a reflection group may appear as the Weyl group for more than one root system. By replacing certain component root systems of type B_n with those of type C_n, every stretched weight lattice over a given reflection group becomes isomorphic, as an abstract module, to some ordinary weight lattice. (See § 1 and the "note added in proof" of [5].)

MAIN THEOREM. *Assume A is a $\mathbf{Z}$-lattice and $G \subset GL(A)$ is a finite group. Then $\mathbf{C}[A]^G$ is a polynomial ring if and only if G is a reflection group and, for some choice of root system, it becomes a Weyl group with A as its weight lattice.*

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Daniel R. Farkas

Virginia Polytechnic Institute
and State University
Blacksburg, VA 24061
USA

NOTE ADDED IN PROOF: As occasionally happens when a mathematician wanders from his area of expertise, he re-invents the wheel. The appendix (§ 4) can be eliminated by invoking a theorem of Serre [B] to the effect that the fixed ring of a suitably nice regular local ring under the action of a finite group is also regular local if and only if the group acts as a pseudo-reflection group on the tangent space of the original local ring. The fifth section is, to a large extent, implicit in work of Steinberg [C]. A statement closer to mine can be found in [A].

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