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Autor: Farkas, Daniel R.

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§ 4. Appendix

$R = R_{(0)} \oplus R_{(1)} \oplus \ldots$ is a graded k-algebra with $R_{(0)} = k$. Let $\mathfrak{m}$ be the maximal ideal $\sum_{i=1}^{\infty} R_{(i)}$. We assume that $\hat{R}$ is a power series ring in finitely many variables. Obviously $\hat{\mathfrak{m}}$ corresponds to the unique maximal ideal of the power series ring, whence $\hat{R}/\hat{\mathfrak{m}}^d$ is always finite dimensional. Since $\hat{\mathfrak{m}}^d$ is homogeneous, some tail $\prod_{?}^{\infty} R_{(i)}$ must then lie in $\hat{\mathfrak{m}}^d$. It follows that the graded algebra of R for the $\mathfrak{m}$-adic filtration is isomorphic to the graded algebra of $\hat{R}$ for the $\hat{\mathfrak{m}}$-adic filtration. The power series assumption implies that the latter is simply a polynomial ring with the standard grading.

Clearly $\mathfrak{m}^2 \subset \sum_{j=2}^{\infty} R_{(j)}$. Hence $R_{(1)}$ injects into $\mathfrak{m}/\mathfrak{m}^2$. Choose a basis for $R_{(1)}$ over k and extend it to a list of homogeneous elements $x_1, \ldots, x_n$ in $\mathfrak{m}$ whose images constitute a basis for $\mathfrak{m}/\mathfrak{m}^2$. It is generally true for any commutative k-algebra R that when $R/\mathfrak{m} = k$ and when the associated graded ring for the $\mathfrak{m}$-adic filtration is the symmetric algebra on $\mathfrak{m}/\mathfrak{m}^2$, that any basis for $\mathfrak{m}/\mathfrak{m}^2$ pulls back to a set of algebraically independent elements in R. In particular, $x_1, \ldots, x_n$ are algebraically independent.

We use the given grading on R to prove that $R = k[x_1, \ldots, x_n]$. Vacuously, $R_{(0)} \subset k[x_1, \ldots, x_n]$. We have chosen the x_i so that $R_{(1)}$ lies in their span, so $R_{(1)} \subset k[x_1, \ldots, x_n]$. Assume, inductively, that $d \geqslant 1$ and $R_{(s)} \subset k[x_1, \ldots, x_n]$ for all $s \leqslant d$. If $y \in R_{(d+1)}$ then

$$y = \Sigma \lambda_i x_i + \Sigma u_j v_j$$

for some $\lambda_i \in k$ and u_j, $v_j \in \mathfrak{m}$. Without loss of generality u_j and v_j are homogeneous and all the x_i and $u_j v_j$ which appear in the formula lie in $\bigcup_{t=1}^{d+1} R_{(t)}$. This can only happen when u_j and v_j are in $R_{(s)}$ for some $s \leqslant d$. By induction, u_j and v_j are elements of $k[x_1, \ldots, x_n]$. Therefore $y \in k[x_1, \ldots, x_n]$.

§ 5. Weyl Groups

It seems to be part of the folklore for Lie theory that the converse of Theorem 8 fails to be true (cf. [4] VI§ 3 Ex. 2). Rather than being dead-ends, these examples serve as inspiration: the machinery of root systems will allow us to determine the correct necessary and sufficient conditions