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$$k(\underline{A})_{\lambda}^G = (k[\underline{A}]^G)^{-1}(k[\underline{A}]_{\lambda}^G).$$

Recall that  $G/C_G(D)$  is a finite group. Hence  $\text{Hom}(G/C_G(D), k^*)$  is finite. Consequently, when  $\text{Hom}(G, k^*)$  is infinite the proposition implies that  $H^1(G, k(\underline{A})^*) \neq 1$ . It is quite plausible (under the assumption  $k^* \cap \underline{A} = 1$ ) that  $H^1(G, k(\underline{A})^*)$  vanishes if and only if  $G$  is finite.

The extra bothersome assumption is vacuous in the case of group algebras. One can read off the following observation from Lemma 2'.

**PROPOSITION 6.** *Assume that  $D = 1$ . Then*

$$1 \rightarrow \text{Hom}(G, k^*) \times H^1(G, A) \rightarrow H^1(G, k(A)^*) \text{ is exact.} \quad \square$$

I have been unable to determine if the injection given by the proposition always splits. Here is one situation where it does.

**PROPOSITION 7.** *Suppose that  $A$  can be fully ordered so that  $G$  acts as a group of order automorphisms of  $A$ . Then the natural map*

$$H^1(G, k^* \cdot A) \rightarrow H^1(G, k(A)^*)$$

splits.

*Proof.* Let  $V: k[\underline{A}] \setminus \{0\} \rightarrow k^* \cdot A$  be the function which sends an element to its “lowest term” with respect to the ordering. The usual degree argument which shows that a polynomial ring is a domain, establishes that  $V$  is multiplicative. Since elements of  $G$  act monotonically,  $V$  is a map of (multiplicative)  $G$ -modules. It is not difficult to check that  $V$  extends to a multiplicative  $G$ -map from  $k(A)^*$  to  $k^* \cdot A$ .

Obviously  $k^* \cdot A \rightarrow k(A)^* \xrightarrow{V} k^* \cdot A$  provides the necessary splitting.  $\square$

The hypothesis of Proposition 7 is very restrictive, even for an infinite cyclic group  $G$ . We leave the following long exercise to the reader. A matrix in  $GL(n, \mathbf{Z})$  is order preserving for some ordering on  $\mathbf{Z}^n$  if and only if each rational irreducible factor of its characteristic polynomial has a positive real root.

### § 3. THE SHEPHARD-TODD-CHEVALLEY THEOREM

Recall that a matrix in  $GL(n, \mathbf{C})$  is a pseudo-reflection if it has finite order and 1 is an eigenvalue of multiplicity  $n - 1$ . The remaining eigenvalue for a pseudo-reflection must be a root of unity; when it is  $-1$  we call

the matrix a reflection. Notice that every pseudo-reflection in  $GL(n, \mathbf{Z})$  must be a reflection. A pseudo-reflection group (resp. reflection group) is a finite group generated by pseudo-reflections (resp. reflections). The classical result is the

**SHEPHARD-TODD-CHEVALLEY THEOREM** (cf. [11], Theorem 4.2.5). *Suppose that  $G$  is a finite group of automorphisms of  $\mathbf{C}[X_1, \dots, X_n]$  which acts linearly. Then  $\mathbf{C}[X_1, \dots, X_n]^G$  is a polynomial ring if and only if  $G$  is a pseudo-reflection group.*

The major theorem of this section is one direction of the STC Theorem for multiplicative actions. Namely,

**THEOREM 8.** *Suppose that  $G \subset GL(n, \mathbf{Z})$  is a finite group of automorphisms of  $A \simeq \mathbf{Z}^n$ . If  $\mathbf{C}[A]^G$  is a polynomial ring then  $G$  is a reflection group.*

This theorem is deduced from the STC Theorem via a connection between abelian group algebras and polynomial rings which goes back to the pioneers of infinite group theory. From now on  $A$  will be the free abelian group on  $n$  generators. Let  $V$  be the  $n$ -dimensional complex vector space  $\mathbf{C} \otimes_{\mathbf{Z}} A$ . If  $x$  is in  $A$  we shall write  $\bar{x} = 1 \otimes x$  in  $V$ . The symmetric algebra on  $V$  will be denoted  $\mathbf{C}[V]$ . (We warn the reader of our primitive tendencies;  $\mathbf{C}[V]$  is not the algebra of polynomial functions on  $V$ .) Both  $\mathbf{C}[A]$  and  $\mathbf{C}[V]$  have canonical augmentations. In the former case the augmentation ideal  $\omega$  is the ideal generated by  $\{x - 1 \mid x \in A\}$ . In the latter,  $\omega$  is the ideal generated by vectors in  $V$ . Let  $\mathbf{C}[A]^\wedge$  and  $\mathbf{C}[V]^\wedge$  be the respective  $\omega$ -adic completions. The exponential function from  $A$  into  $\mathbf{C}[V]^\wedge$  given by

$$\exp(x) = \sum_{j=0}^{\infty} (\bar{x}^j)/j!$$

is well-defined. It extends by linearity and then continuity to a  $\mathbf{C}$ -algebra map  $E: \mathbf{C}[A]^\wedge \rightarrow \mathbf{C}[V]^\wedge$ . In fact,  $E$  is an isomorphism. (The map back extends the logarithm.)

The effect of this identification on automorphisms was first exploited in [1]. A matrix  $g \in GL(A)$  induces an automorphism  $\gamma$  on  $\mathbf{C}[A]^\wedge$ . What is the automorphism after “translating” by  $E$ ? The following calculation of  $E\gamma E^{-1}$  on  $x$  can be checked in detail on the matrix level:

$$\begin{aligned} (E\gamma E^{-1})(\bar{x}) &= E\gamma E^{-1}(\log E(x)) = E(\log^g x) \\ &= E(g(\log x)) = g(\log E(x)) = g(\bar{x}). \end{aligned}$$

**LINEARIZATION THEOREM.** *Let  $G$  be a group of automorphisms of  $A$ , regarded in  $GL(n, \mathbf{Z})$ . Exponentiation extends to an algebra isomorphism  $E: \mathbf{C}[A]^\wedge \rightarrow \mathbf{C}[V]^\wedge$ . Moreover, the multiplicative action of  $G$  (extended by continuity) on  $\mathbf{C}[A]^\wedge$  induces an action on  $\mathbf{C}[V]^\wedge$  which is the extension (by continuity) of the linear action of  $G$  on  $\mathbf{C}[V]$ .*  $\square$

With this tool in hand, the proof of Theorem 8 amounts to carefully keeping track of a myriad of completions and then getting rid of them. The calculations are somewhat clearer in the abstract. So let  $S$  be a  $\mathbf{C}$ -algebra and let  $G$  be a finite group of automorphisms of  $S$ . The averaging or Reynolds operator which sends  $S$  to the fixed ring  $S^G$  is given by

$$av(c) = \frac{1}{|G|} \sum_{g \in G} {}^g c$$

The function  $av$  is an idempotent  $S^G$ -module map.

**LEMMA 9.** *Suppose that  $S$  is a commutative noetherian  $\mathbf{C}$ -algebra and  $I$  is a  $G$ -stable maximal ideal. Then there is a positive integer  $f$  such that*

$$I^{ft} \cap S^G \subset (I \cap S^G)^t \subset I^t \cap S^G \quad \text{for } t = 1, 2, \dots.$$

*Proof.* The second inclusion is obvious. Set  $J = I \cap S^G$ . We first prove that  $I$  is the only prime ideal lying over  $JS$ .

Indeed, suppose  $P$  is a prime ideal of  $S$  containing  $J$ . If  $a \in I$  then  $\prod_{g \in G} {}^g a \in I \cap S^G \subset P$ . By primality there is some  $g \in G$  with  $a \in {}^g P \cap I$ . Consequently,  $I = \bigcup_{g \in G} ({}^g P \cap I)$ , a union of complex subspaces. At least one of these subspaces is not proper: there is an  $h \in G$  such that  $I = {}^h P \cap I$ . Therefore  $I = {}^{h^{-1}} I \subset P$ . Maximality implies  $I = P$ , as required.

The prime radical of  $S/JS$  is the image of  $I$ . But the prime radical is nil and nil ideals in a noetherian ring are nilpotent. Hence there is a positive integer  $f$  such that  $I^f \subset JS$ .

We have established, so far, that  $I^{ft} \subset J^t S$  for all  $t$ . Intersect each side of the inclusion  $S^G$  and apply the averaging operator.

$$I^{ft} \cap S^G = av(I^{ft} \cap S^G) = av(J^t S \cap S^G) \subset av(J^t S) = J^t av(S)$$

We have obtained the necessary inclusion:

$$I^{ft} \cap S^G \subset J^t = (I \cap S^G)^t. \quad \square$$

**LEMMA 10.** *Suppose that  $S$  has a filtration  $S = S_0 \supset S_1 \supset S_2 \supset \dots$  such that each  $S_j$  is  $G$ -stable and  $\cap S_j = 0$ . Then  $(S^\wedge)^G = (S^G)^\wedge$ . (Here*

$S^\wedge$  denotes the completion of  $S$  with respect to the given filtration and  $(S^G)^\wedge$  means the completion of  $S^G$  for the “relative” filtration  $S_j \cap S^G$ .)

*Proof.* There is an obvious injection  $(S^G)^\wedge \rightarrow S^\wedge$ , where the topology on  $(S^G)^\wedge$  coincides with the relative topology on its image. Notice that the action of  $G$  on  $S$  extends continuously to an action on  $S^\wedge$ : if  $a_m \rightarrow a$  then  ${}^g a_m \rightarrow {}^g a$ . It follows that  $(S^G)^\wedge \subset (S^\wedge)^G$ .

Suppose  $b \in (S^\wedge)^G$ . Choose a sequence  $b_m \in S$  such that  $b_m \rightarrow b$ . Then  $\text{av}(b_m) \rightarrow \text{av}(b)$  and  $\text{av}(b) = b$ . Hence  $b \in (S^G)^\wedge$ .  $\square$

LEMMA 11. *Suppose that  $k$  is a field and  $\Phi: k[T_1, \dots, T_n] \rightarrow k$  is a  $k$ -algebra homomorphism. Then there is a change of variables,*

$$k[T_1, \dots, T_n] = k[T'_1, \dots, T'_n],$$

so that  $\ker \Phi = (T'_1, \dots, T'_n)$ .

*Proof.* Consider the automorphism induced by sending each  $T_j$  to  $T'_j = T_j - \Phi(T_j)$ .  $\square$

The next lemma is undoubtedly routine for the expert in commutative algebra. Rather than interrupt the flow of the narrative, we will state it now and then relegate a sketchy proof to the appendix.

DECOMPLETION LEMMA. *Let  $k$  be a field and suppose  $R = R_{(0)} \oplus R_{(1)} \oplus \dots$  is a graded  $k$ -algebra with  $R_{(0)} = k$ . If  $\hat{R}$  (its completion with respect to the grade filtration) is algebra isomorphic to a power series ring  $k[[T_1, \dots, T_n]]$  then  $R$  is isomorphic to a polynomial ring in  $n$  homogeneous variables.*

*Proof of Theorem 8* that if  $\mathbf{C}[A]^G$  is a polynomial ring then  $G$  is a reflection group: According to Lemma 10,  $(\mathbf{C}[A]^\wedge)^G = (\mathbf{C}[A]^G)^\wedge$ . Here  $(\mathbf{C}[A]^G)^\wedge$  is the completion of  $\mathbf{C}[A]^G$  with respect to the filtration  $\omega^t \cap \mathbf{C}[A]^G$ . A straightforward Cauchy sequence argument in conjunction with Lemma 9 shows that  $(\mathbf{C}[A]^G)^\wedge$  is also the  $(\omega \cap \mathbf{C}[A]^G)$ -adic completion. Now  $\mathbf{C}[A]^G$  is a polynomial ring in  $n = \text{rank } A$  variables and  $\omega \cap \mathbf{C}[A]^G$  is a codimension one ideal. By Lemma 11, the  $(\omega \cap \mathbf{C}[A]^G)$ -adic completion of  $\mathbf{C}[A]^G$  is isomorphic to the power series ring  $\mathbf{C}[[T_1, \dots, T_n]]$ .

In summary,  $(\mathbf{C}[A]^\wedge)^G \simeq \mathbf{C}[[T_1, \dots, T_n]]$ . Next, apply the isomorphism  $E$  and use Lemma 10 for the symmetric algebra. We find that  $(\mathbf{C}[V]^G)^\wedge \simeq \mathbf{C}[[T_1, \dots, T_n]]$ . This time,  $\mathbf{C}[V]^G$  is a *graded* algebra under the grading inherited from  $\mathbf{C}[V]$  and its completion is with respect to the grade filtration.

We are in the situation of the Decompletion Lemma for  $\mathbf{C}[V]^G = R$ . Thus  $\mathbf{C}[V]^G$  is a polynomial ring in  $n$  homogeneous variables. Our theorem now follows from the STC Theorem.  $\square$

It is possible to object to the appropriateness of proving a theorem which determines when the invariants for a group algebra comprise a polynomial algebra. After all, the most well-behaved group is the group of order one and its fixed ring is the group algebra we began with. Let's say that a  $\mathbf{C}$ -algebra is an *extended polynomial ring* if it contains algebraically independent elements  $U_1, \dots, U_m, T_1, \dots, T_n$  such that the algebra is isomorphic to  $\mathbf{C}[U_1, U_1^{-1}, \dots, U_m, U_m^{-1}, T_1, \dots, T_n]$ . Equivalently, an extended polynomial ring has the form  $\mathbf{C}[U] \otimes_{\mathbf{C}} \mathbf{C}[T_1, \dots, T_n]$  where  $U$  is a finitely generated free abelian group. Once the generators  $U_i$  and  $T_j$  are distinguished, its augmentation ideal  $\omega$  is the ideal generated by  $U_1 - 1, \dots, U_m - 1, T_1, \dots, T_n$ .

The theorem we have proved can be adapted to prove the “correct” result.

**THEOREM 8<sup>+</sup>.** *Suppose  $G$  is a finite group acting faithfully and multiplicatively on  $\mathbf{C}[A]$ . If  $\mathbf{C}[A]^G$  is an extended polynomial ring then  $G$  is a reflection group.*

*Proof.* We follow the argument a few lines up. It is still true that  $(\mathbf{C}[A])^G$  is the  $(\omega \cap \mathbf{C}[A]^G)$ -adic completion of  $\mathbf{C}[A]^G$ . This time  $\omega \cap \mathbf{C}[A]^G$  is a codimension one ideal in the extended polynomial ring  $\mathbf{C}[A]^G$ . We need Lemma 11<sup>+</sup>: if

$$\Phi: k[U_1^{\pm 1}, \dots, U_m^{\pm 1}, T_1, \dots, T_n] \rightarrow k$$

is an algebra homomorphism then there is a change of variables so that  $\ker \Phi$  becomes the augmentation ideal. (Indeed, define  $U'_j = \Phi(U_j)^{-1}U_j$  and  $T'_j = T_j - \Phi(T_j)$ .)

What is the completion of an extended polynomial ring with respect to powers of its augmentation ideal? Topological abstract nonsense shows that it coincides with  $\mathbf{C}[U]^{\wedge}[[T_1, \dots, T_n]]$  where  $\mathbf{C}[U]^{\wedge}$  is the completion of the group algebra with respect to the  $(U_1 - 1, \dots, U_m - 1)$ -adic topology. But the linearizing  $E$ -isomorphism exhibits  $\mathbf{C}[U]^{\wedge}$  as a power series ring in rank  $U$  variables. In summary, the augmentation-adic completion of an extended polynomial ring is also a power series ring.

From here on, the previous argument can be carried over verbatim.  $\square$

It is much more difficult to decide when  $\mathbf{C}(A)^G$  is a rational function field. The little that is known is surveyed in [7].