

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 30 (1984)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** MULTIPLICATIVE INVARIANTS  
**Autor:** Farkas, Daniel R.  
**Kapitel:** §1. The Bergman-Roseblade Theorem  
**DOI:** <https://doi.org/10.5169/seals-53825>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 27.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## § 1. THE BERGMAN-ROSEBLADE THEOREM

We introduce some notation. Let  $A$  denote a finitely generated free abelian group. Let  $k$  be a field and have  $\underline{A}$  designate an isomorphic copy of  $A$  inside the multiplicative group of some field extension of  $k$ . We will write  $k[\underline{A}]$  for the  $k$ -algebra generated by  $\underline{A}$  and  $k(\underline{A})$  for its field of fractions. The reader is cautioned that  $k[\underline{A}]$  is not the group algebra; distinct elements of  $\underline{A}$  need not be linearly independent over  $k$ . (It's even possible for  $\underline{A}$  to be contained in  $k^*$ .)

We will reserve the notation  $k[A]$ , without the underbar, for the group algebra. There is an obvious relation between the free object  $k[A]$  and  $k[\underline{A}]$ . Indeed, the given isomorphism  $A \simeq \underline{A}$  induces a  $k$ -algebra isomorphism  $k[A]/P \simeq k[\underline{A}]$  where  $P$  is a prime ideal. The ideal  $P$  is "faithful with respect to  $A$ ."

Suppose that  $G$  is a group which acts on  $A$ . Set

$$D = \{a \in A \mid a \text{ has a finite } G\text{-orbit}\}.$$

It is sometimes called the orbital subgroup or relative finite conjugate subgroup.

We are primarily interested in a group  $G$  which acts as a group of  $k$ -automorphisms of  $k[\underline{A}]$ . (The slight awkwardness of language allows us to include possibly nonfaithful actions.) We say that  $G$  acts multiplicatively on  $k[\underline{A}]$  if  $G$  stabilizes  $\underline{A}$ . Thus if  $k[\underline{A}] = k[A]/P$  as described above, then  $P$  is a  $G$ -stable ideal under the corresponding action on  $k[A]$ .

The fundamental theorem in multiplicative invariant theory is Roseblade's Theorem D ([10]). Roseblade based his arguments on profound insights of G. Bergman ([3]).

**BERGMAN-ROSEBLADE THEOREM.** *Assume that  $G$  acts multiplicatively on  $k[\underline{A}] = k[A]/P$ . Then  $P = (P \cap k[D])k[A]$ .*

To understand the implications of this theorem, we take a closer look at  $D$ . Obviously  $D$  is a finitely generated abelian group. Since each generator has a finite  $G$ -orbit,  $D$  is centralized by a subgroup of finite index in  $G$ . In other words,  $G$  acts like a finite group of automorphisms when restricted to  $D$ .

It is easy to see that if a power of an element in  $A$  has a finite orbit, then so does the original element. Hence there is a splitting  $A = D \times B$ . (Unfortunately, there may be no choice of  $B$  which is stabilized by  $G$ .) The

conclusion of the Bergman-Roseblade Theorem can be rewritten—every element in  $P$  has a unique representation  $\sum f(b)b$  where  $b \in B$  and  $f(b) \in P \cap k[D]$ . Thus  $k[A]$  is the group ring  $(k[D])[B]$  for a finitely generated free abelian group  $B$ .

Roseblade proves that the fixed ring  $(k[A])^G$  lies in  $k[D]$  ([10], Lemma 10). This will also be a consequence of the first lemma in the next section. In any event, it has a remarkable consequence.

**THEOREM 1.** *Assume that  $G$  is an arbitrary group acting multiplicatively on  $k[A]$ . Then  $k[A]^G$  is finitely generated.*

*Proof.* As we have remarked,  $(k[A])^G = (k[D])^G$ . But  $G$  acts like a finite group of automorphisms on the affine algebra  $k[D]$ . Noether's Theorem ([11]), states that, in this case, the algebra of invariants is a finitely generated algebra.  $\square$

This is an unexpected surprise. In contrast to the situation for linear actions, Hilbert's 14th problem holds for multiplicative actions without any restriction on the group!

The theme of the paper has emerged. A theory of invariants for multiplicative actions is ultimately a theory for finite groups.

## § 2. GALOIS THEORY

We begin this section by establishing an analogue to the "finiteness" phenomenon of the previous section, for a multiplicative action of  $G$  on  $k(A)$ . Notation is taken from § 1.

**LEMMA 2.** *Suppose that  $G$  acts multiplicatively on  $k(A)$ . Then  $k(A)^G \subset k(D)$ .*

*Proof.* The crucial fact is that  $k(D)[B]$  is a unique factorization domain. If  ${}^g f = f$  for  $f \in k(A)$  then we can write  $f = \alpha/\beta$  where  $\alpha$  and  $\beta$  in  $k(D)[B]$  have no common factors. The invariance of  $f$  becomes

$$({}^g \alpha)\beta = ({}^g \beta)\alpha \quad \text{for all } g \in G.$$

Hence  $\alpha \mid {}^g \alpha$  and  ${}^g \alpha \mid \alpha$ ; we have  $({}^g \alpha)\alpha^{-1}$  a unit in  $k(D)[B]$ . A similar result holds for  $\beta$ .

$${}^g \alpha = u(g)\alpha \quad \text{and} \quad {}^g \beta = w(g)\beta$$

for  $u(g), w(g) \in k(D)^* \cdot B$ .