Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 30 (1984)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: LINEAR ALGEBRA PROOF OF CLIFFORD'S THEOREM

Autor: Gordon, W. J.

Kapitel: 3. Clifford's Theorem — The elementary proof

DOI: https://doi.org/10.5169/seals-53822

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

Download PDF: 16.10.2025

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

THEOREM (Riemann-Roch). Let C be a projective nonsingular algebraic curve. The genus of C is a nonnegative integer g. For all divisors D on C,

$$\dim |D| \geqslant \deg D - g.$$

If the strict inequality holds, D is special. For all special divisors D, $\dim |D| = \deg D + 1 - g + \dim |\mathcal{K} - D|.$

COROLLARY. $\deg \mathcal{K} = 2g - 2$; $\dim |\mathcal{K}| = g - 1$; and all divisors D of degree > 2g - 2 are nonspecial.

3. CLIFFORD'S THEOREM — THE ELEMENTARY PROOF

Clifford's Theorem complements Riemann-Roch by providing information about special divisors, which of necessity are of small degree. The theorem also gives a sufficient condition that the curve C is hyperelliptic. (The theorem owes its name to the appearance of its first part in [1].) The proof I give here is elementary; more typical modern proofs [e.g. 4, Ch. IV, section 5 and 3, Ch. 2, section 3] involve considering whether the canonical morphism $C \to \mathbf{P}_{g-1}$ defined by the canonical divisor \mathcal{K} is an embedding.

Definition. C is a hyperelliptic curve if its genus g is at least 2, and if C admits a $g_{\frac{1}{2}}$.

Remarks.

- 1. C is hyperelliptic if and only if there is a rational map $C \to \mathbf{P}_1$ of degree 2.
- 2. This happens if and only if C has an (affine) equation of the form $y^2 = f(x)$.
- 3. Part (3) of Clifford's Theorem shows that a hyperelliptic curve has a unique g_2^1 . Contrast this to the case of an elliptic curve, where g=1. Here any divisor of degree 2 defines a g_2^1 . Yet choosing distinct points P, Q one sees easily that the divisors 2P and P+Q are not linearly equivalent, and so define distinct g_2^1 's.

THEOREM (Clifford). Let C be a curve of genus g, and let D be an effective special divisor on C. Then

$$(1) \quad \dim |D| \leqslant \frac{1}{2} \deg D.$$

- (2) Equality holds in only 3 cases: (a) D = 0; or
 - (b) $D = \mathcal{K}$; or
 - (c) C is a hyperelliptic curve.
- (3) If Case 2c holds then C admits a unique g_2^1 , $\deg D = 2r$ for some integer $r \ge 1$, and $D \sim r \cdot g_2^1$.

Proof of (1). Since D is effective special, the vector spaces L(D) and $L(\mathcal{K}-D)$ are both of positive dimension. Define a map $\mu: L(D) \times L(\mathcal{K}-D) \to L(\mathcal{K})$ by $\mu(f,g) = f \cdot g$. (Since (f) + D > 0 and $(g) + \mathcal{K} - D > 0$, $(fg) + \mathcal{K} = (f) + (g) + \mathcal{K} = [(f) + D] + [(g) + \mathcal{K} - D] > 0$ so $fg \in L(\mathcal{K})$.) This map is bi-injective, so dim $L(\mathcal{K}) \ge \dim L(D) + \dim L(\mathcal{K}-D) - 1$ by Clifford's Lemma. Since $l(D) = \dim |D| - 1$, one has

(1)
$$\dim |\mathcal{K}| \geqslant \dim |D| + \dim |\mathcal{K} - D|.$$

On the other hand, Riemann-Roch guarantees that

(2)
$$\deg D + 1 - g = \dim |D| - \dim |\mathcal{K} - D|$$
.

Adding these, and recalling that $\dim |\mathcal{K}| = g - 1$, one gets $\deg D \ge 2 \dim |D|$.

Implicit in the proof is a result I will need later.

LEMMA 1. For the effective special divisor D, $\dim |D| = \frac{1}{2} \deg D$ if and only if $\dim |\mathcal{K}| = \dim |D| + \dim |\mathcal{K} - D|$. This holds if and only if $g-1 \leq \dim |D| + \dim |\mathcal{K} - D|$. Further, equality holds for D if and only if it holds for (any effective divisor linearly equivalent to) $\mathcal{K} - D$.

Proof of (2). Assume that equality holds, and that D is neither 0 nor \mathcal{K} . Notice that if deg D=2, or deg $\mathcal{K}-D=2$, then D, or $\mathcal{K}-D$, defines a g_2^1 and C is hyperelliptic. Thus, I may assume that deg D and deg $\mathcal{K}-D$ are both at least 4, so dim |D| and dim $|\mathcal{K}-D|$ are both at least 2. Fix a point P in C. Since dim $|\mathcal{K}-D| \ge 2$ I can choose a divisor $E=P+\Sigma e_R R$ in $|\mathcal{K}-D|$. Now fix a point Q on C but not in the support of E (i.e. $e_Q=0$). Because dim $|D| \ge 2$ I can choose a divisor (sloppily I call it D) in |D| whose support contains both P and Q,

$$D = P + Q + \Sigma d_R R.$$

Set
$$I = \inf(D, E)$$
 and $S = \sup(D, E)$. Then

$$I = \sum \min (d_P, e_P) \cdot P$$
 and $S = \sum \max (d_P, e_P) \cdot P$.

Since P is in I, and Q is not, we have $0 < \deg I < \deg D$. Once I show that dim $|I| = \frac{1}{2} \deg I$, by descent I will have shown that C is hyperelliptic.

Notice that $L(I) = L(D) \cap L(E)$. The inclusion $L(I) \subset L(D) \cap L(D)$ holds because I < D and I < E. On the other hand, if $f \in L(D) \cap L(E)$, (f) + D and (f) + E are both effective. Then, for all points R, $\operatorname{ord}_R(f) \ge -d_R$ and $\operatorname{ord}_R(f) \ge -e_R$, so $\operatorname{ord}_R(f) + \min(d_R, e_R) \ge 0$ and $f \in L(I)$. Similarly, one sees that $L(D) + L(E) \subset L(S)$. Since D < S and E < S both L(D) and L(E) are subspaces of L(S). If $\delta \in L(D)$ and $\varepsilon \in L(E)$, then for all R, $\operatorname{ord}_R(\delta + \varepsilon) \ge \min(\operatorname{ord}_R(\delta), \operatorname{ord}_R(\varepsilon)) \ge \min(-d_R, -e_R) = -\max(d_R, e_R)$. This shows that $\delta + \varepsilon \in L(S)$.

As subspaces of L(S), we see that

$$\dim L(D) + \dim L(E) = \dim L(I) + \dim (L(D) + L(E)).$$

Rewriting this in terms of linear systems gives

$$\dim |D| + \dim |E| \leq \dim |I| + \dim |S|.$$

Since $E \sim \mathcal{K} - D$, Lemma 1 applied to D gives

$$\dim | \mathcal{K} | \leq \dim |I| + \dim |S|.$$

Yet $I + S = D + E \sim \mathcal{K}$, so $S \sim \mathcal{K} - I$. Lemma 1, now applied to I, shows that dim $|I| = \frac{1}{2} \deg I$.

To prove the third part of the theorem I need some technical lemmas. We may assume that the curve C is hyperelliptic and so comes equipped with a given g_2^1 . On any such curve I can define a function $\pi: C \to C$, by defining $\pi(P)$ to be the unique point Q such that P + Q is a divisor in the given g_2^1 . To verify that $\pi(P)$ is well defined, notice that if P + Q and P + R both belong to the given g_2^1 , then $Q \sim R$. Since g > 0, Q must equal R [4, II. 6.10.1]; this shows that $\pi(P)$ is well-defined. Notice that since $\pi P + P$ is in the g_2^1 , $\pi(\pi P) = P$.

LEMMA 2. For any point $P, L(\mathcal{K}-P) = L(\mathcal{K}-P-\pi P)$ and $l(\mathcal{K}-P) = l(\mathcal{K}) - 1$.

Proof. $P + \pi(P)$ is a g_2^1 so dim $|P + \pi P| = 1$ and by Lemma 1, $1 + \dim |\mathcal{K} - P - \pi P| = \dim |\mathcal{K}|$. Since $\mathcal{K} - P - \pi P < \mathcal{K} - P$

 $<\mathscr{K}$, one sees that $L(\mathscr{K}-P-\pi P)\subset L(\mathscr{K}-P)\subset L(\mathscr{K})$. To prove $L(\mathscr{K}-P)=L(\mathscr{K}-P-\pi P)$ it suffices to show that $L(\mathscr{K}-P)\neq L(\mathscr{K})$. Yet if these were equal, the divisor P would be an effective special divisor of degree 1 with $\dim |\mathscr{K}-P|=\dim |\mathscr{K}|$. By Lemma 1, then $\dim |P|$ would equal $\frac{1}{2}\deg P$, which is absurd!

Definition. The points P_1 , ..., P_k on C form a disjoint set of points if for each i, $P_i \neq \pi(P_i)$ and if the divisors $P_i + \pi P_i$ are pairwise disjoint.

LEMMA 3. Let $\{P_1, ..., P_n\}$ be a disjoint set of points, with $n \leq g$. Then

$$\dim \bigcap_{1}^{n} L(\mathcal{K}-P_{i}) = l(\mathcal{K}) - n = g - n.$$

Proof. Since $l(\mathcal{K}-P_i)=l(\mathcal{K})-1$, the intersection has dimension $\geq l(\mathcal{K})-n$. Choose points P_{n+1} , ..., P_g such that $\{P_1,...,P_g\}$ is a disjoint set. Then

$$\bigcap_{1}^{g} L(\mathcal{K} - P_{i}) = \bigcap_{1}^{g} L(\mathcal{K} - P_{i} - \pi P_{i}) = L(\mathcal{K} - \sum_{1}^{g} (P_{i} + \pi P_{i})).$$

If dim $\bigcap_{i=1}^{n} L(\mathcal{K} - P_i) > l(\mathcal{K}) - n$, then

$$\dim L(\mathcal{K}-\Sigma(P_i+\pi P_i)) = \dim \bigcap_{1}^{g} L(\mathcal{K}-P_i) > l(\mathcal{K}) - g = 0.$$

This shows that there is an effective divisor $E \sim \mathcal{K} - \Sigma(P_i + \pi P_i)$; but this is impossible since $\deg (\mathcal{K} - \Sigma(P_i + \pi P_i)) < 0$.

COROLLARY. Let $\{P_1, P_3, ..., P_n\}$ be disjoint. Then

$$\dim (L(\mathcal{K}-2P_1) \bigcap_{3}^{n} L(\mathcal{K}-P_i)) = g - n.$$

Proof. Since $L(\mathcal{K}-2P_1) \subset L(\mathcal{K}-P_1)$, by the lemma $L(\mathcal{K}-2P_1) \cap \bigcap_{i=1}^{n} L(\mathcal{K}-P_i)$ is contained in the vector space $L(\mathcal{K}-P_1) \cap \bigcap_{i=1}^{n} L(\mathcal{K}-P_i)$ of dimension g-n+1.

If these vector spaces were equal, then they would both equal

$$L(\mathcal{K}-2P_1-\pi P_1) \cap \bigcap_{i=1}^{n} L(\mathcal{K}-P_i-\pi P_i)$$
.

Choosing more points P_{n+1} , ..., P_g as in the proof of the lemma would give, similarly,

$$\dim L(\mathcal{K}-2P_1-\pi P_1) \bigcap \bigcap_{3}^{g} L(\mathcal{K}-P_i-\pi P_i) \geqslant 1.$$

Again, we get a contradiction since this shows that the divisor $\mathcal{K} - 2P_1 - \pi P_1 - \sum_{i=3}^{g} (P_i - \pi P_i)$ of negative degree is linearly equivalent to an effective divisor.

Now I can finally prove (3).

Proof of (3). Given an effective special divisor D of degree 2r and with $\dim |D| = r$, choose points P_1 , ..., P_r forming a disjoint set. Notice that since $2 \le \deg D$ and $2 \le \deg (\mathcal{K} - D)$, then $1 \le r \le g - 2$. Then there is a divisor, call it D, in |D| of the form

$$D = P_1 + ... + P_r + A.$$

I claim $A = \pi P_1 + ... + \pi P_r$. This could fail in two ways.

Case 1: If A contains some point Q which is not equal to any of $P_1, ..., P_r$ or $\pi P_1, ..., \pi P_r$, then $L(\mathcal{K} - D) \subset \bigcap_{1}^{r} L(\mathcal{K} - P_i) \cap L(\mathcal{K} - Q)$. Yet $l(\mathcal{K} - D) = \dim |\mathcal{K} - D| + 1 = g - r$ while, by Lemma 3, the intersection has dimension g - (r+1). This shows that Case 1 cannot occur.

Case 2: If A contains some P_i , or contains some πP_i twice, (after interchanging P_i and πP_i if necessary and renumbering) we can write

$$D = 2P_1 + P_2 + ... + P_r + B$$

where B is effective, of degree r-1. Here, $L(\mathcal{K}-D) \subset L(\mathcal{K}-2P_1) \cap L(\mathcal{K}-P_i)$. Again, $l(\mathcal{K}-D) = g-r$, and by the corollary the dimension of the intersection is g-(r+1). Case 2 cannot occur either.

Thus, $D \sim P_1 + ... + P_r + \pi P_1 + ... + \pi P_r$ so $D \sim r \cdot g_2^1$. In particular, if D is any divisor on C of degree 2 with dim |D| = 1, D is linearly equivalent to a divisor in the given g_2^1 . Thus a hyperelliptic curve has a unique g_2^1 .

It is interesting to compare the results of Clifford's theorem with those of the Riemann-Roch theorem, for hyperelliptic curves. Clifford's theorem shows that any special effective divisor D with dim $|D| = \frac{1}{2} \deg D$ is linearly

equivalent to a multiple of the unique g_2^1 . In particular, for the canonical divisor $\mathscr K$ we have $\mathscr K \sim (g-1) \cdot g_2^1$. Conversely, the Riemann-Roch theorem shows that any divisor $D \sim r \cdot g_2^1$, where $1 \leqslant r \leqslant g-1$, satisfies dim $|D| = \frac{1}{2} \deg D$. To see this, note that the proof of part (3) shows that if $D \sim r \cdot g_2^1$ I can write

$$D \sim (P_1 + \pi P_1) + (P_2 + \pi P_2) + \dots + (P_r + \pi P_r)$$

for a disjoint set of points $\{P_1, ..., P_r\}$. Then

$$L(\mathcal{K}-D) = L(\mathcal{K}-\sum_{i=1}^{r} (P_i+\pi P_i)) = \bigcap_{1}^{r} L(\mathcal{K}-P_i).$$

By lemma 3 this set has dimension g-r; in other words, dim $|\mathcal{K}-D|$ = $g-r-1=\frac{1}{2}\deg{(\mathcal{K}-D)}$. By lemma 1, dim $|D|=\frac{1}{2}\deg{D}$.

REFERENCES

- [1] CLIFFORD, William Kingdon. On the Classification of Loci. XXXIII in *Collected Papers*, London, Macmillan & Co., 1882.
- [2] Fulton, William. Algebraic Curves. Reading, MA, Addison-Wesley, 1969.
- [3] Griffiths, Philipp and Joseph Harris. Principles of Algebraic Geometry. New York, John Wiley & Sons, 1978.
- [4] HARTSHORNE, Robin. Algebraic Geometry. New York, Springer-Verlag, 1977.
- [5] WALKER, Robert J. Algebraic Curves. New York, Dover, 1962.

(Reçu le 2 juin 1983)

William J. Gordon

Department of Mathematics State University of New York at Buffalo Buffalo, New York 14214