

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 30 (1984)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** LINEAR ALGEBRA PROOF OF CLIFFORD'S THEOREM  
**Autor:** Gordon, W. J.  
**Kapitel:** 2. A BRIEF RESUME OF DIVISORS ON CURVES  
**DOI:** <https://doi.org/10.5169/seals-53822>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 12.03.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

$$\sigma((\alpha_1, \dots, \alpha_r), (\beta_1, \dots, \beta_s)) = (\alpha_1\beta_1, \dots, \alpha_1\beta_s, \alpha_2\beta_1, \dots, \alpha_2\beta_s, \dots, \alpha_r\beta_s)$$

is a projective morphism establishing an isomorphism between  $\mathbf{P}_{r-1} \times \mathbf{P}_{s-1}$  and the image  $\mathcal{S} = \sigma(\mathbf{P}_{r-1} \times \mathbf{P}_{s-1})$ . [4, Ex I.2.14] Once I label the coordinates of  $\mathbf{P}_{rs-1}$  as  $(z_{11}, \dots, z_{1s}, z_{21}, \dots, z_{2s}, \dots, z_{rs})$ ,  $\mathcal{S}$  can be identified with the algebraic subset of  $\mathbf{P}_{rs-1}$  cut out by the polynomials

$$\{z_{ij}z_{pq} - z_{iq}z_{pj} \mid 1 \leq i, p \leq r \text{ and } 1 \leq j, q \leq s\}.$$

$\mathcal{S}$  is an algebraic subvariety of  $\mathbf{P}_{rs-1}$ , of dimension  $r + s - 2$ .

In  $\mathbf{P}_{rs-1}$  we can also consider the algebraic subvariety  $\mathcal{T}$  cut out by the polynomials  $\{\sum_{ij} z_{ij}\lambda_k^{ij} \mid 1 \leq k \leq t\}$ . Since  $\mathcal{T}$  is cut out by  $t \leq r + s - 2$  equations and  $\dim \mathcal{S} = r + s - 2$ ,  $\mathcal{S}$  and  $\mathcal{T}$  have a nonempty intersection, all of whose components have dimension at least  $(r + s - 2) - t$ , which is  $\geq 0$ . [4, p. 48] However, any intersection point of  $\mathcal{S}$  and  $\mathcal{T}$  corresponds to a pair of points  $(\alpha_1, \dots, \alpha_r) \in \mathbf{P}_{r-1}$ ,  $(\beta_1, \dots, \beta_s) \in \mathbf{P}_{s-1}$  satisfying (\*). The corresponding points  $a = \sum \alpha_i a_i \in A$ ,  $b = \sum \beta_j b_j \in B$  are nonzero, yet  $\varphi(a, b) = 0$ . Since this contradicts the bi-injectivity of  $\varphi$ , I have shown that

$$\dim C \geq r + s - 1. \quad \square$$

The assumption that  $K$  is algebraically closed was only needed to guarantee that  $\mathcal{S} \cap \mathcal{T}$ , which by dimension theory corresponds locally to a proper ideal, was nonempty. Hilbert's Nullstellensatz shows that any proper ideal in a polynomial ring over an algebraically closed field cuts out at least one point.

## 2. A BRIEF RESUME OF DIVISORS ON CURVES

In this section, I will establish notation for divisors, and state the Riemann-Roch theorem. Let  $C$  be a nonsingular projective algebraic curve defined over an algebraically closed field  $K$ .  $C$  is contained in some projective space  $\mathbf{P}_N$  over  $K$ , and a (closed) *point* of  $C$  is any closed point  $(p_0, \dots, p_N)$  of  $\mathbf{P}_N$  at which all the polynomials cutting out  $C$  vanish. The *group of divisors on  $C$*  is the free abelian group generated by the points of  $C$ . Any divisor can be written in the form

$$N = \sum n_p \cdot P$$

where the  $n_p$  are integers, almost all zero. The *degree* of  $N$  is the integer  $\deg N = \sum n_p$ . The divisor  $N$  is *effective* if all the  $n_p$  are  $\geq 0$ ; this is written as  $N \succ 0$ . I write  $D \succ E$  to mean  $D - E \succ 0$ .

To any function  $f$  on  $C$  one can associate a divisor  $(f) = \sum \text{ord}_P(f) \cdot P$ , where  $\text{ord}_P(f)$  is the order of zero or pole of  $f$  at  $P$ . For any function  $f$ , the divisor  $(f)$  has degree 0. The divisors  $D, E$  are *linearly equivalent*, denoted by  $D \sim E$ , if for some function  $f$ ,  $D - E = (f)$ . To a divisor  $D$  on  $C$  one can associate a set of functions on  $C$ ,

$$L(D) = \{\text{functions } f \text{ on } C \mid (f) + D \succ 0\} \cup \{0\}.$$

Then  $L(D)$  is a  $K$ -vector space of dimension  $l(D)$ ; the set  $|D| = \{\text{divisors } E \sim D \mid E \succ 0\}$  of the divisors  $(f) + D$  corresponding to functions  $f$  in  $L(D)$  is the *linear system* associated to  $D$ . If  $\{f_0, \dots, f_n\}$  is a basis of  $L(D)$ , then  $|D|$  can be identified with  $\mathbf{P}_n$  by associating the divisor

$$(a_0 f_0 + \dots + a_n f_n) + D$$

to the triple  $(a_0, \dots, a_n)$ ; one writes  $\dim |D|$  for the dimension of this projective space. To define  $\dim |D|$  intrinsically, notice that  $\dim |D| \geq r$  if and only if, for all points  $P_1, \dots, P_r$  in  $C$ , there is a divisor  $E$  in  $|D|$  of the form  $E = P_1 + \dots + P_r + Q$ , with  $Q$  effective. Any such divisor  $E$  is necessarily effective and linearly equivalent to  $D$ , and has support containing each  $P_i$ . (In fact, since  $\dim |D| \geq r$  there is a linearly independent set  $\{f_0, \dots, f_r\}$  of functions in  $L(D)$ . One can choose  $E$  of the form  $E = D + (\alpha_0 f_0 + \dots + \alpha_r f_r)$  for some  $\alpha_0, \dots, \alpha_r \in K$ .)

If  $D \sim E$ , then  $|D| = |E|$ , so  $\dim |D| = \dim |E|$ , and  $L(D)$  is isomorphic to  $L(E)$ . Since for any function  $f$  on  $C$   $\deg(f) = 0$ , also  $\deg D = \deg E$ . In particular, if  $\deg D < 0$  then  $|D|$  is empty, and  $L(D) = (0)$ .

*Definition.* The curve  $C$  admits a  $g_n^r$  if there exists a divisor  $D$  on  $C$  of degree  $n$ , and with  $\dim |D| = r$ . We call  $|D|$  the  $g_n^r$  defined by  $D$ .

Notice that if  $D$  defines a  $g_n^r$  and  $E \sim D$ , then  $E$  defines the same  $g_n^r$ . Yet a curve may admit several distinct  $g_n^r$ 's if it contains non-linearly equivalent divisors all defining  $g_n^r$ 's. To explain the notation, assume that  $L(D)$  has basis  $(f_0, \dots, f_r)$ . Then the map

$$P \rightarrow (f_0(P), \dots, f_r(P))$$

is a rational map from  $C$  into  $\mathbf{P}_r$ , defined except at the common zeros of all the  $f_i$  (the "fixed points" of  $|D|$ ); via this map, the pullback of every hyperplane in  $\mathbf{P}_r$  is a divisor on  $C$  of degree  $n$ . [4, II: 7.7 and 7.8.1]

The Riemann-Roch Theorem defines for each curve two invariants—a nonnegative integer  $g$ , the *genus*, and a divisor  $\mathcal{K}$ , the *canonical divisor* (determined only up to linear equivalence). [For a modern proof, cf. 4, Ch. IV.1; an elementary proof is given in 2].

**THEOREM (Riemann-Roch).** *Let  $C$  be a projective nonsingular algebraic curve. The genus of  $C$  is a nonnegative integer  $g$ . For all divisors  $D$  on  $C$ ,*

$$\dim |D| \geq \deg D - g.$$

*If the strict inequality holds,  $D$  is special. For all special divisors  $D$ ,*

$$\dim |D| = \deg D + 1 - g + \dim |\mathcal{K} - D|.$$

**COROLLARY.**  $\deg \mathcal{K} = 2g - 2$ ;  $\dim |\mathcal{K}| = g - 1$ ; *and all divisors  $D$  of degree  $> 2g - 2$  are nonspecial.*

### 3. CLIFFORD'S THEOREM — THE ELEMENTARY PROOF

Clifford's Theorem complements Riemann-Roch by providing information about special divisors, which of necessity are of small degree. The theorem also gives a sufficient condition that the curve  $C$  is hyperelliptic. (The theorem owes its name to the appearance of its first part in [1].) The proof I give here is elementary; more typical modern proofs [e.g. 4, Ch. IV, section 5 and 3, Ch. 2, section 3] involve considering whether the canonical morphism  $C \rightarrow \mathbf{P}_{g-1}$  defined by the canonical divisor  $\mathcal{K}$  is an embedding.

*Definition.*  $C$  is a *hyperelliptic curve* if its genus  $g$  is at least 2, and if  $C$  admits a  $g \frac{1}{2}$ .

*Remarks.*

1.  $C$  is hyperelliptic if and only if there is a rational map  $C \rightarrow \mathbf{P}_1$  of degree 2.

2. This happens if and only if  $C$  has an (affine) equation of the form  $y^2 = f(x)$ .

3. Part (3) of Clifford's Theorem shows that a hyperelliptic curve has a unique  $g \frac{1}{2}$ . Contrast this to the case of an elliptic curve, where  $g = 1$ . Here any divisor of degree 2 defines a  $g \frac{1}{2}$ . Yet choosing distinct points  $P, Q$  one sees easily that the divisors  $2P$  and  $P + Q$  are not linearly equivalent, and so define distinct  $g \frac{1}{2}$ 's.

**THEOREM (Clifford).** *Let  $C$  be a curve of genus  $g$ , and let  $D$  be an effective special divisor on  $C$ . Then*