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$$\sigma((\alpha_1, \dots, \alpha_r), (\beta_1, \dots, \beta_s)) = (\alpha_1\beta_1, \dots, \alpha_1\beta_s, \alpha_2\beta_1, \dots, \alpha_2\beta_s, \dots, \alpha_r\beta_s)$$

is a projective morphism establishing an isomorphism between  $\mathbf{P}_{r-1} \times \mathbf{P}_{s-1}$  and the image  $\mathcal{S} = \sigma(\mathbf{P}_{r-1} \times \mathbf{P}_{s-1})$ . [4, Ex I.2.14] Once I label the coordinates of  $\mathbf{P}_{rs-1}$  as  $(z_{11}, \dots, z_{1s}, z_{21}, \dots, z_{2s}, \dots, z_{rs})$ ,  $\mathcal{S}$  can be identified with the algebraic subset of  $\mathbf{P}_{rs-1}$  cut out by the polynomials

$$\{z_{ij}z_{pq} - z_{iq}z_{pj} \mid 1 \leq i, p \leq r \text{ and } 1 \leq j, q \leq s\}.$$

$\mathcal{S}$  is an algebraic subvariety of  $\mathbf{P}_{rs-1}$ , of dimension  $r + s - 2$ .

In  $\mathbf{P}_{rs-1}$  we can also consider the algebraic subvariety  $\mathcal{T}$  cut out by the polynomials  $\{\sum_{ij} z_{ij}\lambda_k^{ij} \mid 1 \leq k \leq t\}$ . Since  $\mathcal{T}$  is cut out by  $t \leq r + s - 2$  equations and  $\dim \mathcal{S} = r + s - 2$ ,  $\mathcal{S}$  and  $\mathcal{T}$  have a nonempty intersection, all of whose components have dimension at least  $(r + s - 2) - t$ , which is  $\geq 0$ . [4, p. 48] However, any intersection point of  $\mathcal{S}$  and  $\mathcal{T}$  corresponds to a pair of points  $(\alpha_1, \dots, \alpha_r) \in \mathbf{P}_{r-1}$ ,  $(\beta_1, \dots, \beta_s) \in \mathbf{P}_{s-1}$  satisfying (\*). The corresponding points  $a = \sum \alpha_i a_i \in A$ ,  $b = \sum \beta_j b_j \in B$  are nonzero, yet  $\varphi(a, b) = 0$ . Since this contradicts the bi-injectivity of  $\varphi$ , I have shown that

$$\dim C \geq r + s - 1. \quad \square$$

The assumption that  $K$  is algebraically closed was only needed to guarantee that  $\mathcal{S} \cap \mathcal{T}$ , which by dimension theory corresponds locally to a proper ideal, was nonempty. Hilbert's Nullstellensatz shows that any proper ideal in a polynomial ring over an algebraically closed field cuts out at least one point.

## 2. A BRIEF RESUME OF DIVISORS ON CURVES

In this section, I will establish notation for divisors, and state the Riemann-Roch theorem. Let  $C$  be a nonsingular projective algebraic curve defined over an algebraically closed field  $K$ .  $C$  is contained in some projective space  $\mathbf{P}_N$  over  $K$ , and a (closed) *point* of  $C$  is any closed point  $(p_0, \dots, p_N)$  of  $\mathbf{P}_N$  at which all the polynomials cutting out  $C$  vanish. The *group of divisors on  $C$*  is the free abelian group generated by the points of  $C$ . Any divisor can be written in the form

$$N = \sum n_p \cdot P$$

where the  $n_p$  are integers, almost all zero. The *degree* of  $N$  is the integer  $\deg N = \sum n_p$ . The divisor  $N$  is *effective* if all the  $n_p$  are  $\geq 0$ ; this is written as  $N \geq 0$ . I write  $D > E$  to mean  $D - E \geq 0$ .

To any function  $f$  on  $C$  one can associate a divisor  $(f) = \sum \text{ord}_P(f) \cdot P$ , where  $\text{ord}_P(f)$  is the order of zero or pole of  $f$  at  $P$ . For any function  $f$ , the divisor  $(f)$  has degree 0. The divisors  $D, E$  are *linearly equivalent*, denoted by  $D \sim E$ , if for some function  $f$ ,  $D - E = (f)$ . To a divisor  $D$  on  $C$  one can associate a set of functions on  $C$ ,

$$L(D) = \{\text{functions } f \text{ on } C \mid (f) + D \succ 0\} \cup \{0\}.$$

Then  $L(D)$  is a  $K$ -vector space of dimension  $l(D)$ ; the set  $|D| = \{\text{divisors } E \sim D \mid E \succ 0\}$  of the divisors  $(f) + D$  corresponding to functions  $f$  in  $L(D)$  is the *linear system* associated to  $D$ . If  $\{f_0, \dots, f_n\}$  is a basis of  $L(D)$ , then  $|D|$  can be identified with  $\mathbf{P}_n$  by associating the divisor

$$(a_0 f_0 + \dots + a_n f_n) + D$$

to the triple  $(a_0, \dots, a_n)$ ; one writes  $\dim |D|$  for the dimension of this projective space. To define  $\dim |D|$  intrinsically, notice that  $\dim |D| \geq r$  if and only if, for all points  $P_1, \dots, P_r$  in  $C$ , there is a divisor  $E$  in  $|D|$  of the form  $E = P_1 + \dots + P_r + Q$ , with  $Q$  effective. Any such divisor  $E$  is necessarily effective and linearly equivalent to  $D$ , and has support containing each  $P_i$ . (In fact, since  $\dim |D| \geq r$  there is a linearly independent set  $\{f_0, \dots, f_r\}$  of functions in  $L(D)$ . One can choose  $E$  of the form  $E = D + (\alpha_0 f_0 + \dots + \alpha_r f_r)$  for some  $\alpha_0, \dots, \alpha_r \in K$ .)

If  $D \sim E$ , then  $|D| = |E|$ , so  $\dim |D| = \dim |E|$ , and  $L(D)$  is isomorphic to  $L(E)$ . Since for any function  $f$  on  $C$   $\deg(f) = 0$ , also  $\deg D = \deg E$ . In particular, if  $\deg D < 0$  then  $|D|$  is empty, and  $L(D) = (0)$ .

*Definition.* The curve  $C$  admits a  $g_n^r$  if there exists a divisor  $D$  on  $C$  of degree  $n$ , and with  $\dim |D| = r$ . We call  $|D|$  the  $g_n^r$  defined by  $D$ .

Notice that if  $D$  defines a  $g_n^r$  and  $E \sim D$ , then  $E$  defines the same  $g_n^r$ . Yet a curve may admit several distinct  $g_n^r$ 's if it contains non-linearly equivalent divisors all defining  $g_n^r$ 's. To explain the notation, assume that  $L(D)$  has basis  $(f_0, \dots, f_r)$ . Then the map

$$P \rightarrow (f_0(P), \dots, f_r(P))$$

is a rational map from  $C$  into  $\mathbf{P}_r$ , defined except at the common zeros of all the  $f_i$  (the "fixed points" of  $|D|$ ); via this map, the pullback of every hyperplane in  $\mathbf{P}_r$  is a divisor on  $C$  of degree  $n$ . [4, II: 7.7 and 7.8.1]

The Riemann-Roch Theorem defines for each curve two invariants—a nonnegative integer  $g$ , the *genus*, and a divisor  $\mathcal{K}$ , the *canonical divisor* (determined only up to linear equivalence). [For a modern proof, cf. 4, Ch. IV.1; an elementary proof is given in 2].

**THEOREM (Riemann-Roch).** *Let  $C$  be a projective nonsingular algebraic curve. The genus of  $C$  is a nonnegative integer  $g$ . For all divisors  $D$  on  $C$ ,*

$$\dim |D| \geq \deg D - g.$$

*If the strict inequality holds,  $D$  is special. For all special divisors  $D$ ,*

$$\dim |D| = \deg D + 1 - g + \dim |\mathcal{K} - D|.$$

**COROLLARY.**  $\deg \mathcal{K} = 2g - 2$ ;  $\dim |\mathcal{K}| = g - 1$ ; *and all divisors  $D$  of degree  $> 2g - 2$  are nonspecial.*

### 3. CLIFFORD'S THEOREM — THE ELEMENTARY PROOF

Clifford's Theorem complements Riemann-Roch by providing information about special divisors, which of necessity are of small degree. The theorem also gives a sufficient condition that the curve  $C$  is hyperelliptic. (The theorem owes its name to the appearance of its first part in [1].) The proof I give here is elementary; more typical modern proofs [e.g. 4, Ch. IV, section 5 and 3, Ch. 2, section 3] involve considering whether the canonical morphism  $C \rightarrow \mathbf{P}_{g-1}$  defined by the canonical divisor  $\mathcal{K}$  is an embedding.

*Definition.*  $C$  is a *hyperelliptic curve* if its genus  $g$  is at least 2, and if  $C$  admits a  $g_2^1$ .

*Remarks.*

1.  $C$  is hyperelliptic if and only if there is a rational map  $C \rightarrow \mathbf{P}_1$  of degree 2.

2. This happens if and only if  $C$  has an (affine) equation of the form  $y^2 = f(x)$ .

3. Part (3) of Clifford's Theorem shows that a hyperelliptic curve has a unique  $g_2^1$ . Contrast this to the case of an elliptic curve, where  $g = 1$ . Here any divisor of degree 2 defines a  $g_2^1$ . Yet choosing distinct points  $P, Q$  one sees easily that the divisors  $2P$  and  $P + Q$  are not linearly equivalent, and so define distinct  $g_2^1$ 's.

**THEOREM (Clifford).** *Let  $C$  be a curve of genus  $g$ , and let  $D$  be an effective special divisor on  $C$ . Then*