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Artikel: LINEAR ALGEBRA PROOF OF CLIFFORD'S THEOREM
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$$\sigma((\alpha_1, ..., \alpha_r), (\beta_1, ..., \beta_s)) = (\alpha_1\beta_1, ..., \alpha_1\beta_s, \alpha_2\beta_1, ..., \alpha_2\beta_s, ..., \alpha_r\beta_s)$$

is a projective morphism establishing an isomorphism between $\mathbf{P}_{r-1} \times \mathbf{P}_{s-1}$ and the image $\mathcal{S} = \sigma(\mathbf{P}_{r-1} \times \mathbf{P}_{s-1})$. [4, Ex I.2.14] Once I label the coordinates of $\mathbf{P}_{rs-1}$ as $(z_{11}, ..., z_{1s}, z_{21}, ..., z_{2s}, ..., z_{rs})$, $\mathcal{S}$ can be identified with the algebraic subset of $\mathbf{P}_{rs-1}$ cut out by the polynomials

$$\{z_{ij}\, z_{pq} - z_{iq}z_{pj} \mid 1 \leqslant i, p \leqslant r \text{ and } 1 \leqslant j, q \leqslant s\}.$$

$\mathcal{S}$ is an algebraic subvariety of $\mathbf{P}_{rs-1}$, of dimension $r + s - 2$.

In $\mathbf{P}_{rs-1}$ we can also consider the algebraic subvariety $\mathcal{T}$ cut out by the polynomials $\{\sum_{ij} z_{ij}\lambda_k^{ij} \mid 1 \leqslant k \leqslant t\}$. Since $\mathcal{T}$ is cut out by $t \leqslant r + s - 2$ equations and $\dim \mathcal{S} = r + s - 2$, $\mathcal{S}$ and $\mathcal{T}$ have a nonempty intersection, all of whose components have dimension at least $(r+s-2) - t$, which is $\geqslant 0$. [4, p. 48] However, any intersection point of $\mathcal{S}$ and $\mathcal{T}$ corresponds to a pair of points $(\alpha_1, ..., \alpha_r) \in \mathbf{P}_{r-1}, (\beta_1, ..., \beta_s) \in \mathbf{P}_{s-1}$ satisfying (*). The corresponding points $a = \Sigma\, \alpha_i a_i \in A$, $b = \Sigma\, \beta_j b_j \in B$ are nonzero, yet $\varphi(a, b) = 0$. Since this contradicts the bi-injectivity of φ, I have shown that

$$\dim C \geqslant r + s - 1. \qquad \square$$

The assumption that K is algebraically closed was only needed to guarantee that $\mathcal{S} \cap \mathcal{T}$, which by dimension theory corresponds locally to a proper ideal, was nonempty. Hilbert's Nullstellensaltz shows that any proper ideal in a polynomial ring over an algebraically closed field cuts out at least one point.

2. A brief resume of divisors on curves

In this section, I will establish notation for divisors, and state the Riemann-Roch theorem. Let C be a nonsingular projective algebraic curve defined over an algebraically closed field K. C is contained in some projective space $\mathbf{P}_N$ over K, and a (closed) *point* of C is any closed point $(p_0, ..., p_N)$ of $\mathbf{P}_N$ at which all the polynomials cutting out C vanish. The *group of divisors on* C is the free abelian group generated by the points of C. Any divisor can be written in the form

$$N = \Sigma\, n_P \cdot P$$

where the n_P are integers, almost all zero. The *degree* of N is the integer $\deg N = \Sigma\, n_P$. The divisor N is *effective* if all the n_P are $\geqslant 0$; this is written as $N \succ 0$. I write $D \succ E$ to mean $D - E \succ 0$.

To any function f on C one can associate a divisor $(f) = \Sigma \operatorname{ord}_P(f) \cdot P$, where $\operatorname{ord}_P(f)$ is the order of zero or pole of f at P. For any function f, the divisor (f) has degree 0. The divisors D, E are *linearly equivalent*, denoted by $D \sim E$, if for some function f, $D - E = (f)$. To a divisor D on C one can associate a set of functions on C,

$$L(D) = \{\text{functions } f \text{ on } C \mid (f) + D \succ 0\} \bigcup \{0\}.$$

Then $L(D)$ is a K-vector space of dimension $l(D)$; the set $|D| = \{\text{divisors } E \sim D \mid E \succ 0\}$ of the divisors $(f) + D$ corresponding to functions f in $L(D)$ is the *linear system* associated to D. If $\{f_0, ..., f_n\}$ is a basis of $L(D)$, then $|D|$ can be identified with $\mathbf{P}_n$ by associating the divisor

$$(a_0 f_0 + ... + a_n f_n) + D$$

to the triple $(a_0, ..., a_n)$; one writes $\dim |D|$ for the dimension of this projective space. To define $\dim |D|$ intrinsically, notice that $\dim |D| \geqslant r$ if and only if, for all points $P_1, ..., P_r$ in C, there is a divisor E in $|D|$ of the form $E = P_1 + ... + P_r + Q$, with Q effective. Any such divisor E is necessarily effective and linearly equivalent to D, and has support containing each P_i. (In fact, since $\dim |D| \geqslant r$ there is a linearly independent set $\{f_0, ..., f_r\}$ of functions in $L(D)$. One can choose E of the form $E = D + (\alpha_0 f_0 + ... + \alpha_r f_r)$ for some $\alpha_0, ..., \alpha_r \in K$.)

If $D \sim E$, then $|D| = |E|$, so $\dim |D| = \dim |E|$, and $L(D)$ is isomorphic to $L(E)$. Since for any function f on C $\deg (f) = 0$, also $\deg D = \deg E$. In particular, if $\deg D < 0$ then $|D|$ is empty, and $L(D) = (0)$.

Definition. The curve C *admits a* g_n^r if there exists a divisor D on C of degree n, and with $\dim |D| = r$. We call $|D|$ *the* g_n^r *defined by D*.

Notice that if D defines a g_n^r and $E \sim D$, then E defines the same g_n^r. Yet a curve may admit several distinct g_n^r's if it contains non-linearly equivalent divisors all defining g_n^r's. To explain the notation, assume that $L(D)$ has basis $(f_0, ..., f_r)$. Then the map

$$P \to (f_0(P), ..., f_r(P))$$

is a rational map from C into $\mathbf{P}_r$, defined except at the common zeros of all the f_i (the "fixed points" of $|D|$); via this map, the pullback of every hyperplane in $\mathbf{P}_r$ is a divisor on C of degree n. [4, II: 7.7 and 7.8.1]

The Riemann-Roch Theorem defines for each curve two invariants—a nonnegative integer g, the *genus*, and a divisor $\mathscr{K}$, the *canonical divisor* (determined only up to linear equivalence). [For a modern proof, cf. 4, Ch. IV.1; an elementary proof is given in 2].

THEOREM (Riemann-Roch). *Let* C *be a projective nonsingular algebraic curve. The genus of* C *is a nonnegative integer* g. *For all divisors* D *on* C,

$$\dim |D| \geqslant \deg D - g .$$

If the strict inequality holds, D *is* special. *For all special divisors* D,

$$\dim |D| = \deg D + 1 - g + \dim |\mathscr{K} - D| .$$

COROLLARY. $\deg \mathscr{K} = 2g - 2$; $\dim |\mathscr{K}| = g - 1$; *and all divisors* D *of degree* $> 2g - 2$ *are nonspecial.*

3. CLIFFORD'S THEOREM — THE ELEMENTARY PROOF

Clifford's Theorem complements Riemann-Roch by providing information about special divisors, which of necessity are of small degree. The theorem also gives a sufficient condition that the curve C is hyperelliptic. (The theorem owes its name to the appearance of its first part in [1].) The proof I give here is elementary; more typical modern proofs [e.g. 4, Ch. IV, section 5 and 3, Ch. 2, section 3] involve considering whether the canonical morphism $C \to \mathbf{P}_{g-1}$ defined by the canonical divisor $\mathscr{K}$ is an embedding.

Definition. C is a *hyperelliptic* curve if its genus g is at least 2, and if C admits a g^1_2.

Remarks.

1. C is hyperelliptic if and only if there is a rational map $C \to \mathbf{P}_1$ of degree 2.

2. This happens if and only if C has an (affine) equation of the form $y^2 = f(x)$.

3. Part (3) of Clifford's Theorem shows that a hyperelliptic curve has a unique g^1_2. Contrast this to the case of an elliptic curve, where $g = 1$. Here any divisor of degree 2 defines a g^1_2. Yet choosing distinct points P, Q one sees easily that the divisors $2P$ and $P + Q$ are not linearly equivalent, and so define distinct g^1_2's.

THEOREM (Clifford). *Let* C *be a curve of genus* g, *and let* D *be an effective special divisor on* C. *Then*