

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 29 (1983)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** MILNOR LATTICES AND GEOMETRIC BASES OF SOME SPECIAL SINGULARITIES  
**Autor:** Ebeling, Wolfgang  
**Kapitel:** 2. Geometric Bases of the Milnor Lattice  
**DOI:** <https://doi.org/10.5169/seals-52982>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 19.08.2025

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

## 2. GEOMETRIC BASES OF THE MILNOR LATTICE

There are certain classes of bases for the Milnor lattice, which are distinguished by the geometry of the singularity. We shall recall the definition of these bases (see [12] for more information). For that purpose we shall consider the semiuniversal deformation of the singularity.

Let  $X_0 = f^{-1}(0)$  be the complex analytic hypersurface defined by  $f$ . Let  $F: X \rightarrow S$  be a suitable representative of the semi-universal deformation of the germ  $(X_0, 0)$ . Denote by  $D \subset S$  the corresponding discriminant, i.e. the image of the critical locus of  $F$ . Put  $S' = S - D$ ,  $X' = F^{-1}(S')$  and  $F' = F|_{X'}$ . Then  $F': X' \rightarrow S'$  is a  $C^\infty$ -fiber bundle, where each fiber is diffeomorphic to the Milnor fiber.

Choose a generic complex line in the affine space containing  $S$ , which intersects  $S$  in a complex disc  $\Delta$ . Then  $\Delta$  intersects the discriminant  $D$  in  $\mu$  points  $c_1, \dots, c_\mu$ , which lie in the interior of  $\Delta$ . Choose a basepoint  $s_0$  on the boundary of  $\Delta$ . Let  $X_{s_0}$  denote the fiber of  $F$  over  $s_0$ . Then  $H_{n-1}(X_{s_0}, \mathbf{Z})$  is isomorphic to  $L$  and will also be denoted by  $L$ . We shall construct bases of  $L$ . Each path  $\phi_i$  in  $\Delta' = \Delta - \{c_1, \dots, c_\mu\}$  from  $c_i$  to  $s_0$  determines an element of  $L$  as follows. The fiber over  $c_i$  has only one singular point which is an ordinary double point. Near this point, the fibers are given locally by an equation

$$z_1^2 + \dots + z_n^2 = r^2.$$

The real sphere

$$S^{n-1} = \{(x_1, \dots, x_n) \mid x_1^2 + \dots + x_n^2 = r^2\}, \quad z_i = x_i + iy_i,$$

represents after the choice of an orientation a homology class in a fiber over a point of  $\phi_i$  near  $c_i$ . Transport along the path  $\phi_i$  gives an element  $e \in L$  satisfying  $\langle e, e \rangle = -2$ . Such an element is called *vanishing cycle*. Let  $\Lambda^*$  denote the set of vanishing cycles. Choosing a path  $\phi_i$  for each  $c_i$  yields a system of  $\mu$  vanishing cycles.

In order to get a basis of vanishing cycles, there are several possible restrictions on the choice of paths. In order to define these restrictions and the corresponding classes of bases, we need the notion of a *simple loop* corresponding to a path  $\phi_i$  from  $c_i$  to  $s_0$ . This is the element of  $\pi_1(\Delta', s_0)$  represented by the loop  $\tau_i$  going from  $s_0$  to a point sufficiently near to  $c_i$  along the path  $\phi_i$ , going once around  $c_i$  in the positive direction (counterclockwise) and returning to  $s_0$  along the path  $\phi_i$ . This loop induces an automorphism of  $L$ , the Picard-Lefschetz-transformation corresponding to the path  $\phi_i$ . It is given by the reflection  $s_{e_i}$  on the orthogonal complement of the vanishing cycle  $e_i$  corresponding to  $\phi_i$ , i.e.  $s_{e_i}$  is defined by

$$s_{e_i}(x) = x + \langle x, e_i \rangle e_i \quad \text{for } x \in L.$$

1) *Distinguished Bases*

Choose the paths  $\phi_i$  non-selfintersecting, let any two have only  $s_0$  as a common point and number the paths in the order in which they arrive at  $s_0$ , counted clockwise beginning from the boundary of  $\Delta$ . Then the corresponding system of vanishing cycles forms a basis, and a basis of  $L$  obtained in this way is called a *distinguished basis*. Let  $\mathcal{B}^*$  denote the set of all distinguished bases of  $L$ .

On  $\mathcal{B}^*$  there is an operation of the braid group  $Z_\mu$  in  $\mu$  strings, where a generator  $\alpha_i$  operates as follows: Let  $B = \{e_1, \dots, e_\mu\} \in \mathcal{B}^*$  be a distinguished basis defined by the system of paths  $\{\phi_1, \dots, \phi_\mu\}$ . The operation  $\alpha_i$  is induced by the following elementary operation on the level of paths (cf. Fig. 1):

$$(\phi_1, \dots, \phi_\mu) \rightarrow (\phi_1, \dots, \phi_{i-1}, \phi_{i+1}\tau_i, \phi_i, \phi_{i+2}, \dots, \phi_\mu).$$

The operation  $\alpha_i$  is then given by

$$(e_1, \dots, e_\mu) \rightarrow (e_1, \dots, e_{i-1}, s_{e_i}(e_{i+1}), e_i, e_{i+2}, \dots, e_\mu).$$

The inverse operation  $\alpha_i^{-1}$  is also denoted by  $\beta_{i+1}$ .

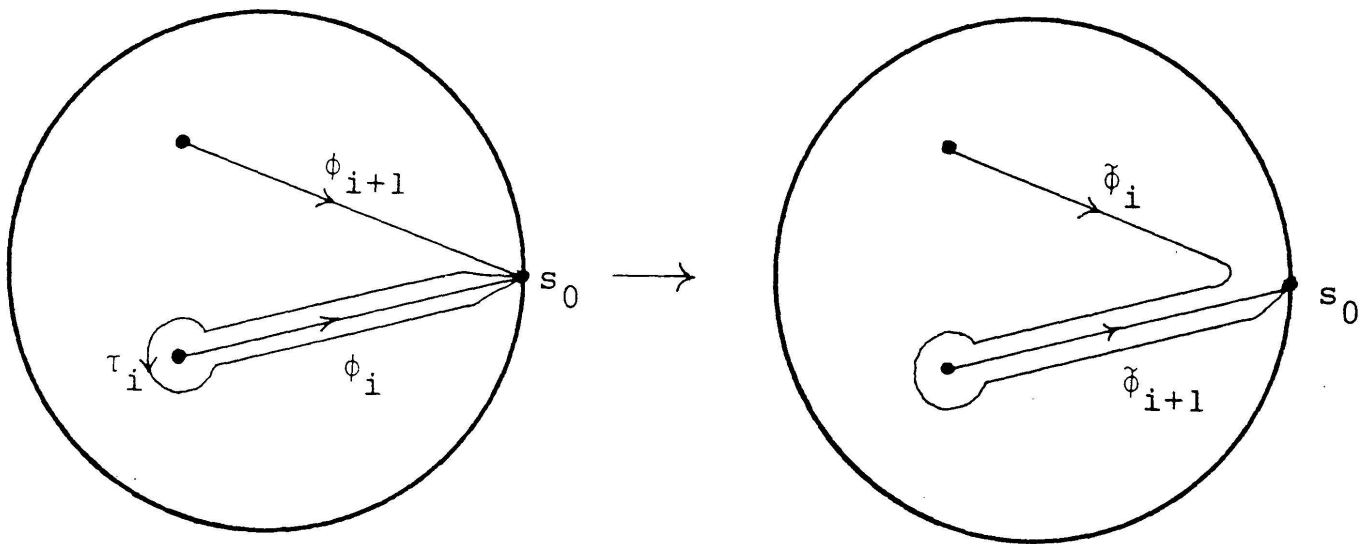


FIGURE 1

A distinguished basis does not only depend on the choice of paths, but also on the choice of orientation of the cycles. The change of orientation of a cycle  $e_i$  corresponds to an operation  $\gamma_i$  given by  $\gamma_i(e_j) = e_j$  for  $j \neq i$  and  $\gamma_i(e_i) = -e_i$ .

Therefore there is in addition an operation of  $(\mathbb{Z}/2\mathbb{Z})^\mu$  on  $\mathcal{B}^*$ . Together one has an operation of the semidirect product

$$Z^* = Z_\mu \rtimes (\mathbb{Z}/2\mathbb{Z})^\mu$$

on  $\mathcal{B}^*$  (both factors considered as subgroups of the symmetric group of  $\mathcal{B}^*$ ).

PROPOSITION 2.1. *The operation of  $Z^*$  on  $\mathcal{B}^*$  is transitive.*

For a proof see [12].

PROPOSITION 2.2. *If  $\mathcal{B}^*$  contains a basis  $B = \{e_1, \dots, e_\mu\}$  with  $\langle e_i, e_j \rangle \in \{0, 1, -1\}$  for  $i \neq j$ , then already  $Z_\mu$  operates transitively on  $\mathcal{B}^*$ .*

*Proof.* It suffices to show that any basis  $\tilde{B} \in \mathcal{B}^*$  can be transformed into  $B$  by  $Z_\mu$ . By the previous proposition there exists an element of  $Z^*$  which transforms  $\tilde{B}$  to  $B$ . We show that for each  $i$  there exists an element  $\sigma_i \in Z_\mu$  such that  $\gamma_i(B) = \sigma_i(B)$ . If  $\langle e_j, e_{j+1} \rangle = \varepsilon$ ,  $\varepsilon = \pm 1$ , then  $\alpha_j^{1/2} = id$  and

$$\alpha_j^{3\varepsilon}(B) = \gamma_{j+1}(B), \quad \alpha_j^{-3\varepsilon}(B) = \gamma_j(B).$$

Now let  $k$  be the smallest integer such that  $\langle e_i, e_{i+k} \rangle \neq 0$ ,  $k \neq 0$ . We consider the case  $k > 0$ , the case  $k < 0$  is analogous. If  $k = 1$ , we apply the previous remark. Otherwise the transformation  $(\alpha_{i+k-2} \circ \dots \circ \alpha_i)(B)$  interchanges  $e_i$  and  $e_{i+k-1}$  and leaves all other basis elements fixed. Hence one can now apply the above remark. This proves the proposition.

Let  $B = \{e_1, \dots, e_\mu\}$  be a distinguished basis. Then

$$h = s_{e_1} \circ s_{e_2} \circ \dots \circ s_{e_\mu}$$

is the *classical monodromy operator* of the singularity.

## 2) Weakly Distinguished Bases

We now impose the only condition on the system of paths that the corresponding simple loops generate  $\pi_1(\Delta', s_0)$ . Then it can be shown that the corresponding system of vanishing cycles forms again a basis, and a basis obtained in this way is called a *weakly distinguished basis*. Let  $\mathcal{B}^0$  denote the set of all weakly distinguished bases.

Since the numbering does not play a role for a weakly distinguished basis, we have on  $\mathcal{B}^0$  also an operation of the symmetric group  $\mathcal{S}_\mu$  of degree  $\mu$ . Let  $Z^0$  be

the group generated by  $Z^*$  and  $\mathcal{S}_\mu$ . There are some special elements in  $Z^0$  defined as follows: Let  $B = \{e_1, \dots, e_\mu\}$  be a weakly distinguished basis defined by a system of paths  $\{\phi_1, \dots, \phi_\mu\}$ . Let  $\{\tau_i\}$  be the corresponding system of simple loops. For  $i \neq j$  we define a transformation  $\alpha_i(j)$  on the different levels as follows:

$$\begin{aligned}(\phi_1, \dots, \phi_\mu) &\rightarrow (\phi_1, \dots, \phi_{j-1}, \phi_j \tau_i, \phi_{j+1}, \dots, \phi_\mu), \\(\tau_1, \dots, \tau_\mu) &\rightarrow (\tau_1, \dots, \tau_{j-1}, \tau_i^{-1} \tau_j \tau_i, \tau_{j+1}, \dots, \tau_\mu), \\(e_1, \dots, e_\mu) &\rightarrow (e_1, \dots, e_{j-1}, s_{e_i}(e_j), e_{j+1}, \dots, e_\mu).\end{aligned}$$

The inverse transformation is denoted by  $\beta_i(j)$ . In the case  $n \equiv 3(4)$  it coincides with  $\alpha_i(j)$  on the homology level. Now  $Z^0$  is generated by the transformations  $\alpha_i(j)$ ,  $\mathcal{S}_\mu$  and change of orientation operations.

CONJECTURE 2.3 (Gusein-Zade). *The operation of  $Z^0$  on  $\mathcal{B}^0$  is transitive.*

This conjecture can be reduced to a problem in pure combinatorial group theory, see [12]. It is not known to the author, whether the conjecture is true.

The *monodromy group*  $\Gamma$  of the singularity  $f$  is the image of  $\pi_1(\Delta', s_0)$  under the natural representation

$$\rho: \pi_1(\Delta', s_0) \rightarrow \text{Aut}(L).$$

It is generated by the reflections corresponding to the elements of a weakly distinguished basis.

The matrix of the bilinear form on  $L$  with respect to a basis  $B$  of vanishing cycles is described in the usual way by a graph with (possibly multiple) edges weighted by  $+1$  or  $-1$ , where we indicate negative weight by a dotted line. This graph is called the *Dynkin diagram* with respect to  $B$ .

### 3. MILNOR LATTICES AND WEAKLY DISTINGUISHED BASES OF SOME SPECIAL SINGULARITIES

We shall consider the Milnor lattices of some specific singularities, namely the singularities of Arnold's lists and the minimally elliptic hypersurface singularities. By the singularities of Arnold's lists we mean the singularities, for which Arnold has given normal forms in [1], i.e. the singularities of the series  $A$ ,  $D$ ,  $J$ ,  $E$ ,  $X$ ,  $Y$ ,  $Z$ ,  $W$ ,  $T$ ,  $Q$ ,  $S$ ,  $U$  and  $V$ . Most of these series contain singularities with arbitrary number of moduli. The minimally elliptic hypersurface singularities can be defined as follows (cf. [5]): They are the singularities  $f$ :