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Briefly the proof of Theorem 5.3 goes as follows: By a standard argument,  $\pi_i(PL/A_k)$  coincides with the concordance classes of  $A_k$ -structures on  $S^i$  (the exotic  $A_k$ -spheres). Since  $\pi_i(PL/A) = \lim_{\leftarrow} \pi_i(PL/A_k)$  it follows by definitions that the inclusion  $\pi_i(PL/A) \rightarrow \eta_i^A$  is an injection, where  $\eta_i^A$  is the cobordism group of  $i$ -dimensional  $A$ -manifolds. Then we construct a Thom space  $MA$  such that  $\pi_i(MA) \approx \eta_i^A$  (by using a transversality argument for  $A$ -manifolds). Then it turns out that the map  $\eta_i^A \rightarrow H_i(B_A; \mathbf{Z}/2\mathbf{Z})$  given by  $\{M \xrightarrow{v_M} B_A\} \mapsto (v_M)_* [M]$  is an injection. We can put these maps into the following commutative diagram:

$$\begin{array}{ccccc} & \pi_i(PL/A) & \rightarrow & \eta_i^A & \\ & \downarrow f & & \downarrow & \\ h \swarrow & & & & \\ H_i(PL/A; \mathbf{Z}) & \xrightarrow{r} & H_i(PL/A; \mathbf{Z}/2\mathbf{Z}) & \xrightarrow{g} & H_i(B_A; \mathbf{Z}/2\mathbf{Z}) \end{array}$$

where  $h$  is the Hurewicz map,  $r$  is the reduction and  $g$  is induced by inclusion. Since the other two maps are injections then  $f$  must be injection. In fact  $f$  is a split injection since it is a map between  $\mathbf{Z}/2\mathbf{Z}$ -vector spaces. Hence  $h$  is a split injection. This implies that all  $k$ -invariants of  $PL/A$  is zero, i.e.  $PL/A$  is a product of Eilenberg-McClaine spaces  $\prod K(\mathbf{Z}/2\mathbf{Z}, n_i)$ . Then by dualizing the split injection  $g \circ f$  we get a surjection

$$H^i(B_A; \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\lambda} \text{Hom}(\pi_i(PL/A); \mathbf{Z}/2\mathbf{Z})$$

Let  $\delta_{n_i} \in H^{n_i}(B_A; \mathbf{Z}/2\mathbf{Z})$  such that  $\lambda(\delta_{n_i})$  is the generator of  $\mathbf{Z}/2\mathbf{Z}$ .

$$\delta = \prod \delta_{n_i} \text{ defines a map } B_A \rightarrow \prod_i K(\mathbf{Z}/2\mathbf{Z}, n_i) = PL/A.$$

Then the map  $\pi \times \delta: B_A \rightarrow B_{PL} \times PL/A$  turns out to be the desired splitting. The calculation of  $\rho_n$  can be done by using the geometric interpretation of  $\pi_*(PL/A)$ .

The set  $\mathcal{S}_A(M) = \bigoplus_n H^n(M; \pi_n(PL/A))$  measures the number of different "topological resolutions" of  $M$ , up to concordance (i.e.  $A$ -structures). Therefore often  $\mathcal{S}_A(M)$  is infinite; and  $\mathcal{S}_A(M^8)$  has  $2^{2^6}$  elements for any closed 8-manifold  $M^8$ .

## §6. ON CLASSIFICATION OF REAL ALGEBRAIC SETS

The resolution and complexification properties of real algebraic sets impose many restrictions on the underlying topological spaces. To give a topological characterization of algebraic sets one has to find all such properties, such that a

set is homeomorphic to an algebraic set if and only if it satisfies these properties. Call a polyhedron  $V$  an *Euler space* if  $\chi(\text{Link}(x))$  is even for all vertices  $x \in V$ . Recall that all algebraic sets are Euler spaces, in fact in low dimensions this topological property completely determines compact algebraic sets (and hence all algebraic sets by Proposition 3.1).

**THEOREM 6.1.** *Let  $X$  be a compact polyhedron of dimensions  $\leq 2$ . Then  $X$  is homeomorphic to a real algebraic set if and only if  $X$  is an Euler space.*

This theorem was announced in [AK<sub>2</sub>] and a proof was given [AK<sub>7</sub>]. Since [AK<sub>7</sub>] did not appear in print we repeat that proof here. This proof is very useful to understand the high dimensional case. It is done by first constructing a “topological resolution” for  $X$  then proceeding as in the proof of Theorem 5.1.

*Proof:* The proof of case  $\dim(X) \leq 1$  follows from Theorem 4.1, so assume that  $\dim(X) = 2$ . Let  $X'$  be the barycentric subdivision of  $X$ . Let  $X_i =$  the  $i$ -skeleton of  $X'$ . Then (exercise)  $X_1$  satisfies the even local Euler characteristic condition also. We will say a one simplex in  $X'$  has type  $i$  ( $i = 0, 1$ ) if the number of faces containing it is congruent to  $2i \pmod{4}$ . Let  $X_{1i}$  be the unions of edges of type  $i$ , then (exercise)  $X_{10}$  and  $X_{11}$  each satisfy the even local Euler characteristic condition. Hence, they have resolutions  $\pi_{1i} : Z_{1i} \rightarrow X_{1i}$  where  $Z_{1i}$  are unions of circles, and the  $\pi_{1i}$  are diffeomorphisms over  $X_{1i} - X_0$ .

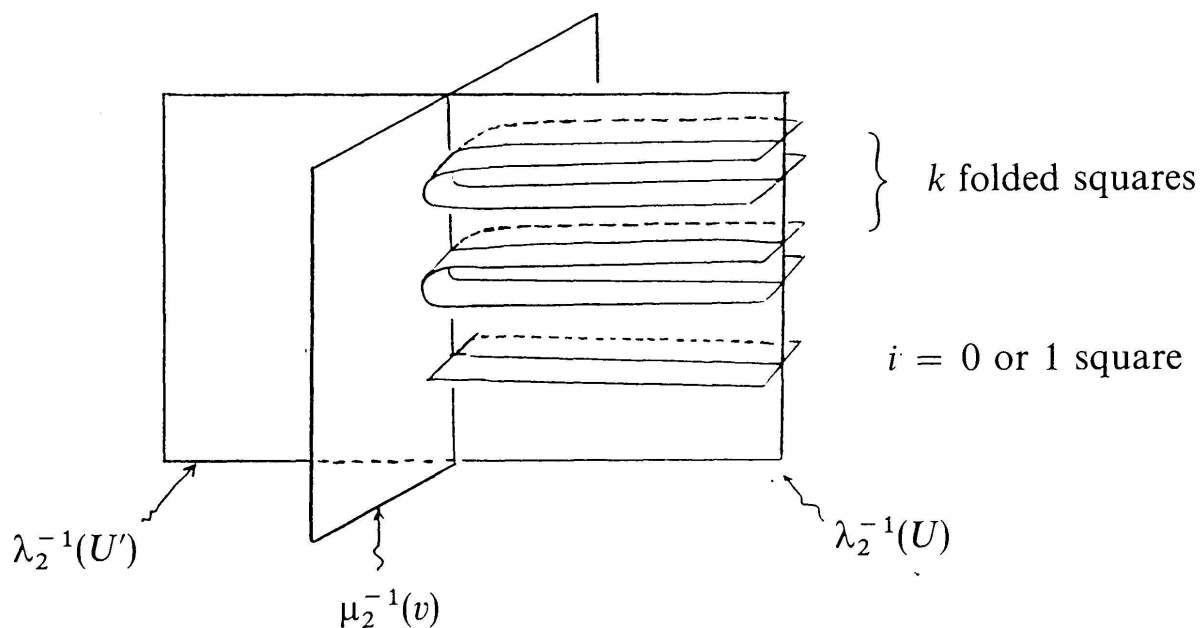
First, we imbed  $X_0$  in  $\mathbf{R}^4$ . Now let  $V_1 = B(\mathbf{R}^4, X_0)$  and let  $\mu_1 : V_1 \rightarrow \mathbf{R}^4$  be the projection. We may imbed  $Z_{10} \cup Z_{11}$  in  $V_1$  so that  $\mu_1(Z_{1i}) \cup X_0$  is homeomorphic to  $X_{1i}$  and  $\mu_1|_{Z_{1i}} = \pi_{1i}$ . Since  $V_1$  has totally algebraic homology, by Theorem 2.8 we may assume after replacing  $V_1$  by  $V_1 \times \mathbf{R}^n$  that each component of each  $Z_{1i}$  is a nonsingular algebraic subset of  $V_1$ . We now let  $V_2 = B(V_1, Z_{10} \cup Z_{11})$  and  $\lambda_2 : V_2 \rightarrow V_1$  be the projection and  $\mu_2 : V_2 \rightarrow \mathbf{R}^4$  be the composition of  $\mu_1$  and  $\lambda_2$ . We will now imbed a surface  $Z_2$  in  $V_2$  so that

$$\mu_2(Z_2) \cup \mu_1(Z_{10} \cup Z_{11}) \cup X_0$$

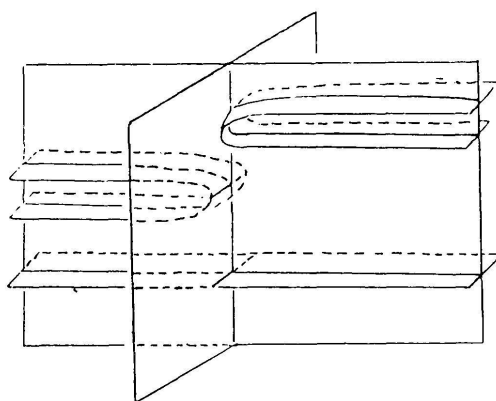
is homeomorphic to  $X$ .

We pick some pairing of the faces coming into each edge, i.e. there are an even number of them, and we divide them into groups of two. This gives a resolution of  $X - X_0$ , namely, take the disjoint union of the faces with vertices deleted and identify two edges if they are in the same group of two. This will be part of our surface  $Z_2$ , but we will not imbed it until later. We will first imbed the part of  $Z_2$  lying over a small neighborhood of  $X_0$ .

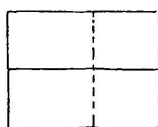
Take any vertex  $v$  of  $X_0$  and let  $e$  be an edge containing  $v$ , let  $i = 0, 1$  be such that  $e \subset X_{1i}$ . Then  $e = \mu_1(U)$  for some interval  $U$  in  $Z_{1i}$ . Let there be  $4k + 2i$  faces containing  $e$ . Pick a point  $p$  in  $\mu_2^{-1}(v) \cap \lambda_2^{-1}(u)$  where  $u \in U$  is the point so that  $\mu_1(u) = v$ . Then in a neighborhood of  $p$ , we have two codimension one submanifolds  $\mu_2^{-1}(v)$  and  $\lambda_2^{-1}(Z_{1i})$ . We imbed  $k + i$  squares in a neighborhood of  $p$  as indicated below.



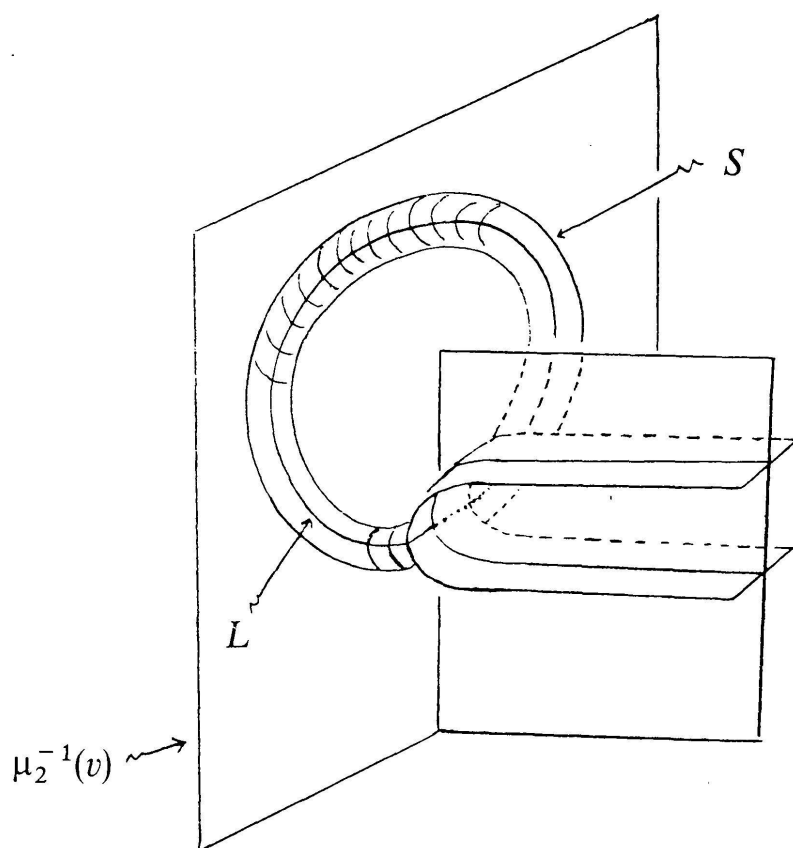
We do this for each edge containing  $v$ . Notice that one of these edges is  $\mu_1(U')$  for some interval  $U'$  in  $Z_{1i}$  so  $U' \cap U = u$ , i.e. the interval on the other side of  $u$ . If  $i = 1$ , we connect the bottom squares of the two sides together as shown below.



In the end, we have a bunch of squares



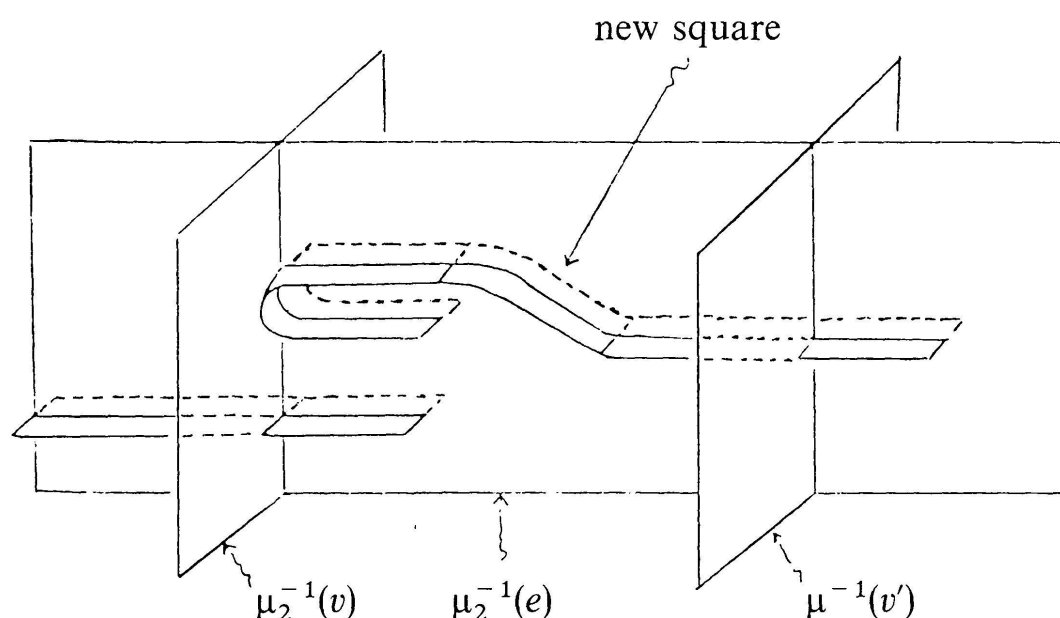
whose horizontal midlines are mapped by  $\mu_2$  to  $v$  and whose vertical midlines are mapped by  $\lambda_2$  to  $Z_{10} \cup Z_{11}$ . Furthermore, this map is either equivalent to  $x^2$  or  $x$  if we choose our imbedding nicely. To each corner of each square, we may assign a face of  $X'$  which contains  $v$  so that the following conditions are met: each face containing  $v$  is assigned to exactly two corners, if  $e$  is the edge containing  $\mu_2$  of the top half of the vertical midline, then the faces assigned to the top two corners each contain  $e$  and are, in fact, paired, and likewise, for the bottom two corners and the bottom midline half. We may now form a number of polygons by taking the vertical side edges of all the squares and identifying their endpoints, if the corresponding faces are the same. We claim these polygons are the boundary of a surface  $S$  which contains  $L$ , a union of arcs and circles in general position so that  $S$  is a regular neighborhood of  $L$ ,  $\partial S \cap L$  is the union of the endpoints of all the arcs in  $L$  and  $\partial S \cap L$  is also the union of all the midpoints of the sides of the boundary polygons.



Given this, we imbed  $S$  in  $V_2$  so that  $S$  misses  $\lambda_2^{-1}(Z_{10} \cup Z_{11})$  and  $\mu_2^{-1}(X_0 - v)$  and so  $\mu_2^{-1}(v) \cap S = L$ , and so  $S$  intersects the squares we have already imbedded in the union of the side edges of all the squares, furthermore, these intersect in the natural way so that the point of  $L \cap \partial S$  which corresponds to the midpoint of a side of a polygon, is mapped to the midpoint of the corresponding side of a square. So, letting  $S'$  be  $S$  union all the squares, we have that  $\mu_2(S')$  is

homeomorphic to the star of  $v$  in the union of the faces of  $X$ . This is because clearly  $\mu_2(S')$  is the cone on  $\mu_2(\partial S')$ , but  $\mu_2(\partial S')$  is obtained by taking the polygon formed by all the top and bottom sides of the squares and identifying endpoints corresponding to the same face and identifying midpoints of all sides which map to the same edge of  $X'$ . This is clearly the link of  $v$  in the closure of all faces.

We do this for all the vertices and we get a surface  $S''$ . We now add some more squares. For each edge  $e$  of  $X'$ , let  $v$  and  $v'$  be its vertices. We have previously paired up the faces containing  $e$ . For each pair of faces, we have a corresponding top or bottom side of a square over  $v$ , and a top or bottom side of a square over  $v'$  (namely the sides between the two corners assigned to the pair), we connect these two sides with another square as shown ( $S$  is not shown).



If we do this for each pair of faces coming into each edge of  $X'$ , we get a surface  $S^*$  imbedded in  $V_2$  so that  $\mu_2(S^*)$  is homeomorphic to a neighborhood of  $X_1$  in the union of the faces of  $X'$ . It is now easy to imbed a bunch of discs (one for each face of  $X'$ ) and so get a surface  $Z_2$  in  $V_2$ , so that  $\mu_2(Z_2)$  is the union of the faces of  $X'$  and so

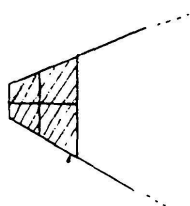
$$\mu_2(Z_2) \cup \mu_1(Z_{10} \cup Z_{11}) \cup X_0$$


is homeomorphic to  $X$ .

We could now try to approximate  $Z_2$  by a nonsingular algebraic set and then blow down to finish off the proof, but the problem is  $Z_2$  is not stable, i.e.  $Z_2$  is not

transverse to  $\mu_2^{-1}(X_0)$ . However, we may, after replacing  $V_2$  by  $V_2 \times \mathbf{R}^k$ , assume that  $Z_2 \cap \mu_2^{-1}(X_0)$  is a union of nonsingular algebraic sets. An exercise below shows that if we blow up along each of these algebraic sets twice, then  $Z_2$  becomes transverse to  $\mu_2^{-1}(X_0)$ . Then we are able to finish off by approximating  $Z_2$  by an algebraic set (Theorem 2.8) and blowing down, first over  $Z_{10} \cup Z_{11}$  and then over  $X_0$  (Proposition 3.3).

We deferred the proof that the polygon bounds the surfaces  $S$ , so we give it here. First, by induction, we may assume all polygons have either one or two sides, for we may take three sides and fill in part of the surface and reduce to the problem with those three sides replaced by one side (see below).

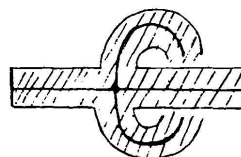


The shaded region is filled in part, + is part of  $L$ . If we can fill in the rest, then adding on  will fill in all of it.

But we can easily fill in a polygon with two sides, and we can also fill in two one sides. Since the total number of sides is even, we are done.



two sides filled in



two one-sides filled in

*Exercise:* Think of  $\mathbf{R}^n$  as  $\{(x, y, z, w) \mid x, y, z \in \mathbf{R} \text{ and } w \in \mathbf{R}^{n-3}\}$ . Let  $S = \{z = x^a y^b, w = 0\}$  and  $T = \{z = 0\}$ . Blow up along the  $x$  axis twice and along the  $y$  axis twice, and show that after blowing up  $S$  becomes transverse to the inverse image of  $T$ , (assuming  $a = 1, 2$  and  $b = 1$  or  $2$ ). Note that by imbedding the  $S$  in the above proof correctly, we may assume that locally it looks like this with  $T = \mu_2^{-1}(v)$ . □

The proof of the 2-dimensional case is done by first constructing an appropriate topological resolution. In the general case this leads us to make the following definition. A *topological resolution tower*  $\{V_i, V_{ji}, p_{ji}\}$  is a collection of smooth manifolds  $V_i, i = 0, \dots, n$ , subsets  $V_{ji} \subset V_i, j = 0, \dots, i - 1$  and maps  $p_{ji} : V_{ji} \rightarrow V_j$  satisfying the following properties:

- (I)  $p_{ji}(V_{ji} \cap V_{ki}) \subset V_{kj}$  for  $k < j < i$ .
- (II)  $p_{kj} \circ p_{ji}|_{V_{ji} \cap V_{ki}} = p_{ki}|_{V_{ji} \cap V_{ki}}$  for  $k < j < i$ .
- (III)  $p_{ji}^{-1}(\bigcup_{m \leq k} V_{mj}) = V_{ji} \cap \bigcup_{m \leq k} V_{mi}$ .
- (IV)  $V_{kj}$  is a union of codimension one smooth submanifolds of  $V_j$  in general position; we call them the sheets of  $V_{kj}$ . If  $S$  is a sheet of  $V_{kj}$  then  $p_{ji}^{-1}(S)$  is the intersection of  $V_{ji}$  with a union of sheets of  $\bigcup_{m \leq k} V_{mi}$ .
- (V)  $p_{ji}$  is smooth on each sheet of  $V_{ji}$ , and

$$p_{ji}: V_{ji} - \bigcup_{k < j} V_{ki} \rightarrow V_j - \bigcup_{k < j} V_{kj}$$

is a locally trivial fibration.

- (VI) For any  $q \in V_{ji}$  let  $q_i = q$ ,  $q_j = p_{ji}(q)$ .

Then there are smooth local coordinates

$$\theta_a: (U_a, 0) \xrightarrow{\sim} (V_a, q_a), \quad a = i, j,$$

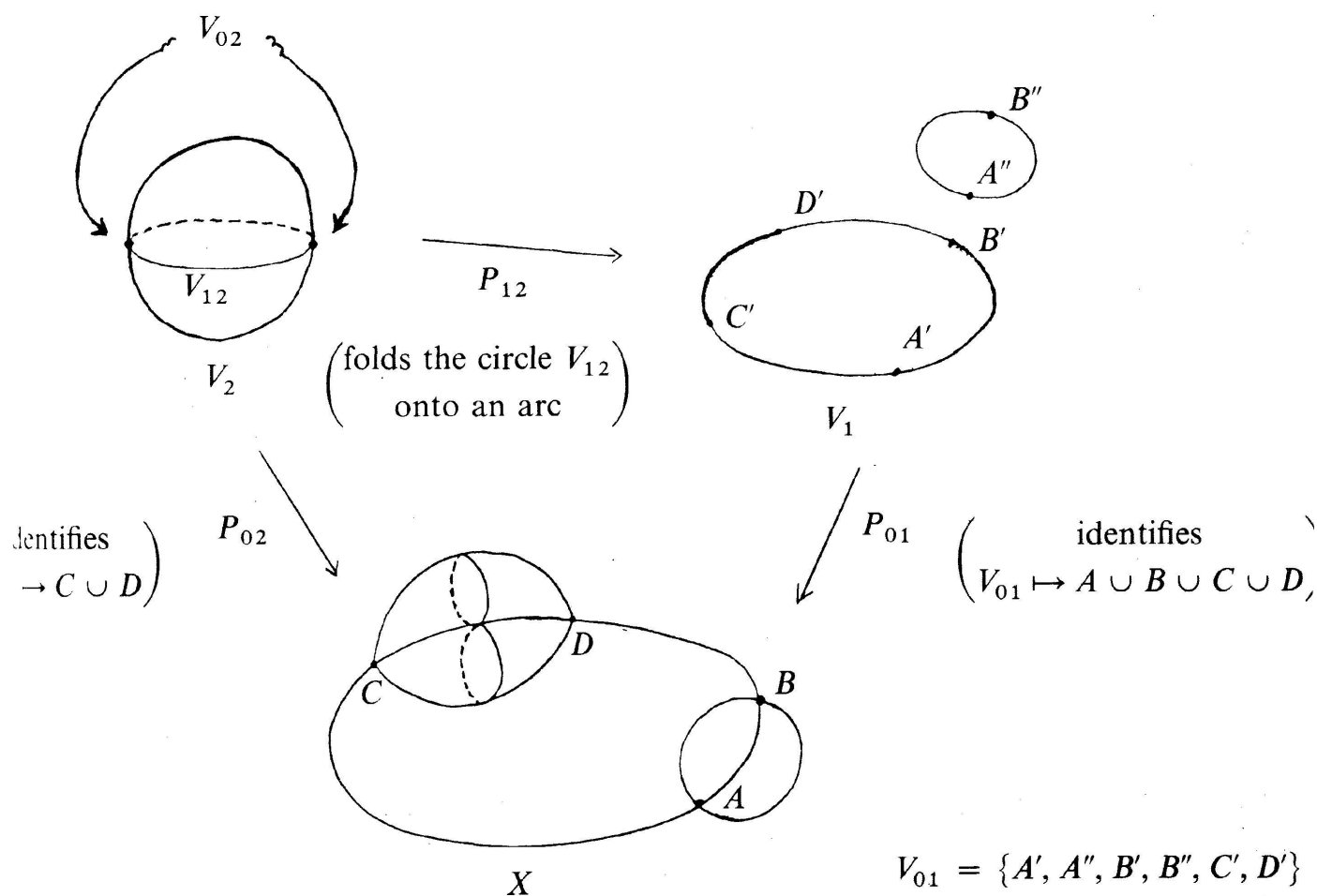
where  $U_a$  is an open neighborhood of 0 in some  $\mathbf{R}^{c_{a0}} \times \mathbf{R}^{c_{a1}} \times \dots \times \mathbf{R}^{c_{aa}}$  such that:

- (1)  $\theta_a^{-1}(V_{ta}) = \begin{cases} \emptyset & \text{if } c_{at} = 0, \\ \{x \mid \prod_{s=1}^{c_{at}} x_{ts} = 0\} \cap U_a & \text{if } c_{at} \neq 0, \end{cases}$
- (2)  $[\theta_j^{-1} \circ p_{ji} \circ \theta_i(x)]_{km} = \prod_{t=0}^k \prod_{s=1}^{c_{it}} x_{ts}^{I_{km}^{ts}} \cdot \phi_{km}(x)$  if  $k < j$ ,

where  $I_{km}^{ts}$  is a nonnegative integer, and each  $\phi_{km}$  is a nowhere zero smooth function.  $x_{ts}$  denotes the  $s$ -th coordinate of  $x$  in  $\mathbf{R}^{c_{it}}$ , and  $[\theta_j^{-1} \circ p_{ji} \circ \theta_i(x)]_{km}$  denotes the  $m$ -th coordinate of  $\theta_j^{-1} \circ p_{ji} \circ \theta_i(x)$  in  $\mathbf{R}^{c_{jk}}$ .

Even though (VI) looks like an algebraic condition it is a topological condition. It says that topologically the map  $p_{ji}$  has only certain types of singularities (i.e. it folds or crushes). We call a topological resolution tower  $\{V_i, V_{ji}, p_{ji}\}$  an *algebraic resolution tower* if all  $V_i, V_{ji}$  are compact algebraic sets and  $p_{ij}$  are entire rational functions.

The realization  $|\mathcal{T}|$  of a (topological or algebraic) resolution tower  $\mathcal{T} = \{V_i, V_{ji}, p_{ji}\}$  is the quotient space  $\bigcup V_i / x \sim p_{ji}(x)$  for  $x \in V_{ji}$ .  $|\mathcal{T}|$  is a stratified space with  $i$ -th stratum equal to  $V_i - \bigcup_{j < i} V_{ji}$ . It turns out that if  $\mathcal{T}$  is an algebraic resolution tower then  $|\mathcal{T}|$  is an algebraic set.  $|\mathcal{T}|$  is a generalization of an  $A$ -space.



Real algebraic sets are obvious candidates for realizations of topological resolution towers: If  $X$  is a real algebraic set, it has an algebraic stratification

$$X_0 \subset X_1 \subset \dots \subset X_{n-1} \subset X_n = X$$

with  $\text{Sing}(X_i) \subset X_{i-1}$ ,  $i = 1, \dots, n$ . Then the resolution of singularities theorem [H] says that there is a multiblowup:

$$V_n = Z_n \xrightarrow{\pi_n} Z_{n-1} \rightarrow \dots \rightarrow Z_1 \xrightarrow{\pi_1} Z_0 = X$$

with  $\pi_1 : Z_1 \rightarrow Z_0$  is a multiblowup of  $X$  which resolves the singularities of  $X_1$ , i.e. there is a nonsingular  $V_1 \subset Z_1$  making the following commute

$$\begin{array}{ccc} V_1 & \hookrightarrow & Z_1 \\ \downarrow & & \downarrow \pi_1 \\ X_1 & \hookrightarrow & Z_0 \end{array}$$

If  $\pi_{ji} : Z_i \rightarrow Z_j$  is the composition projection, then  $\pi_{i+1}$  is a multiblowup of  $Z_i$  which resolves the singularities of the strict preimage of  $X_{i+1}$  under  $\pi_{0i}$ , i.e. there is a nonsingular  $V_{i+1} \subset Z_{i+1}$  and the commutative diagram

$$\begin{array}{ccc} V_{i+1} & \hookrightarrow & Z_{i+1} \\ \downarrow & & \downarrow \pi_{0,i+1} \\ X_{i+1} & \hookrightarrow & Z_0 \end{array}$$

Let  $V_{ji} = \pi_{ji}^{-1}(V_j) \cap V_i$  and  $\pi_{ji}|_{V_{ji}} = p_{ji} : V_{ji} \rightarrow V_j$ . Then one can show that

$$X \approx \bigcup V_i / p_{ji}(x) \sim x,$$

for  $x \in V_{ji}$ .

In fact after refining this process one gets:

**THEOREM 6.2.** *A set is an algebraic set if and only if it is homeomorphic to a realization  $|\mathcal{T}|$  of some algebraic resolution tower  $\mathcal{T} = \{V_i, V_{ji}, p_{ji}\}$ .*

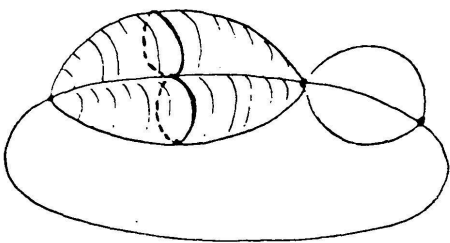
Hence we have natural maps

$$\begin{array}{ccc} \left\{ \text{Algebraic sets} \right\} & \xrightarrow{\tau} & \left\{ \begin{array}{l} \text{realization of topological} \\ \text{resolution towers} \end{array} \right\} \\ \searrow \text{onto} & & \nearrow \rho \\ & \left\{ \begin{array}{l} \text{realization of algebraic} \\ \text{resolution towers} \end{array} \right\} & \end{array}$$

where  $\rho$  is the forgetful map, and  $\tau$  is the composition. We will denote the set of realization of topological resolution towers by  $\mathcal{R}$ . To characterize algebraic sets topologically, we need to show that  $\rho$  maps onto  $\mathcal{R}$ . Presently to prove this we need each  $V_i$  to be diffeomorphic to a nonsingular algebraic set with totally algebraic homology (see §2). We believe that these restrictions should not be necessary.

Once surjectivity of  $\tau$  is proven, then it would be useful to find the combinatorial conditions which characterize elements of  $\mathcal{R}$  (i.e. algebraic sets). For spaces of dimension  $\leq 2$  the only condition is that the space has to be an

Euler space (Theorem 6.1). In dimension 3 this is not sufficient. For example if  $X^3$  is the suspension  $\Sigma(Y^2)$  of  $Y^2$  where

$$Y^2 = \text{[Diagram]} = \Sigma(\text{figure 8}) \cup \Sigma(\text{three points}) \cup \text{an arc}$$


then  $X^3$  is an Euler space but it can not be in  $\mathcal{R}$ ; in particular  $X^3$  can not be homeomorphic to an algebraic set (also see  $[K_2]$  for a discussion of this).

In general we start with a Thom stratified space  $X$ , by refining the stratification we can assume that each stratum has a trivial normal bundle. Then by proceeding as in  $[Su_2]$  we can find obstructions  $\alpha_k \in H^k(X(k); \Gamma_{n-k-1})$  to  $X$  being an algebraic set with this stratification, where  $X(k)$  is the  $k$ -th stratum of  $X$ ,  $n = \dim(X)$  and  $\Gamma_i$  is the cobordism group of  $i$ -dimensional elements of  $\mathcal{R}$ . For example we can show  $\Gamma_0 = \Gamma_1 \cong \mathbf{Z}/2\mathbf{Z}$  and  $\Gamma_2 \cong (\mathbf{Z}/2\mathbf{Z})^{16}$ . It would be useful to compute the cobordism groups  $\Gamma_*$  for  $* \geq 3$  or reduce the computation to a certain homotopy group of a universal space (as in the smooth cobordism group). A more precise discussion of this section will appear in  $[AK_9]$ .