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**Autor:** Akbulut, Selman / King, Henry  
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## §5. ALGEBRAIC STRUCTURES ON P.L. MANIFOLDS

To prove that P.L. manifolds are homeomorphic to algebraic sets we first define a class of stratified spaces ( $A$ -spaces) which admit "topological resolutions" to smooth manifolds, then we prove that these spaces are homeomorphic to algebraic sets. Then the result is achieved by showing that this class is big enough to contain all P.L. manifolds.

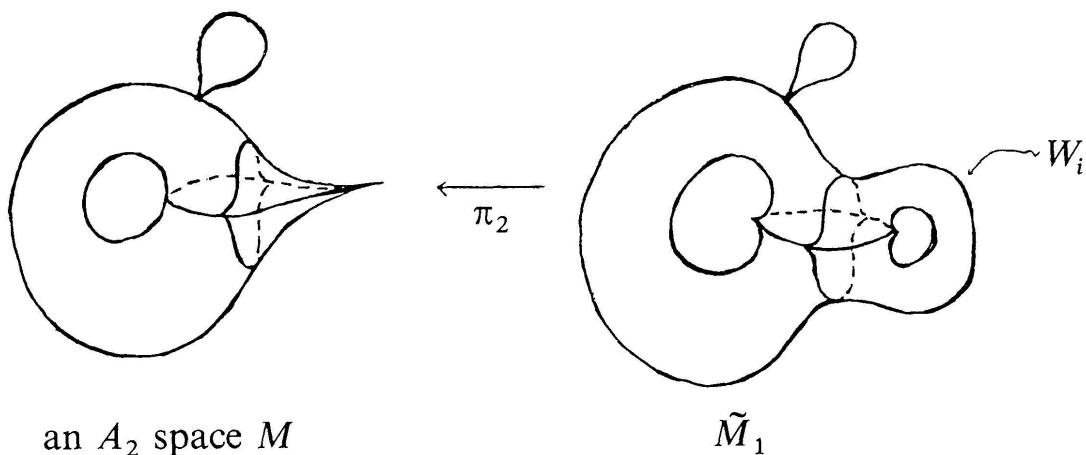
Define  $A_0$ -spaces to be smooth manifolds. Inductively let  $A_k$ -spaces to be spaces in the form  $M = M_0 \bigcup_{\partial} N_i \times \text{cone}(\Sigma_i)$  where  $M_0$  is an  $A_{k-1}$ -space and  $\Sigma_i$  are boundaries of compact  $A_{k-1}$ -spaces and  $N_i$  are smooth manifolds. The union is taken along codimension zero subsets of  $\partial M_0$  and  $N_i \times \Sigma_i \subset N \times \text{cone}(\Sigma_i)$ . We define

$$\partial M = (\partial M_0 - \bigcup N_i \times \Sigma_i) \cup \bigcup \partial N_i \times \text{cone}(\Sigma_i),$$

hence boundaries of  $A_k$ -spaces are  $A_k$ -spaces. We call a space an  $A$ -space if it is an  $A_k$ -space for some  $k$ . If in the above definition we also assume that each  $\Sigma_i$  is a P.L. sphere then we call the resulting  $A$ -space  $A$ -manifold.  $A$ -manifolds are P.L. manifolds equipped with above special structure.  $A$ -spaces are more general than  $A$ -manifolds, for example they don't have to be manifolds.

$A$ -spaces are constructed so that they can be "topologically" resolved. If  $M$  is an  $A_k$ -space  $M_0 \cup \bigcup N_i \times \text{cone}(\Sigma_i)$ , we can choose compact  $A_{k-1}$  spaces  $W_i$  with  $\partial W_i = \Sigma_i$ . We can construct the obvious  $A_{k-1}$  space  $\tilde{M}_{k-1} = M_0 \cup \bigcup N_i \times W_i$ . There is the obvious map  $\pi_k: \tilde{M}_{k-1} \rightarrow M$  which is identity on  $M_0$  and takes  $N_i \times W_i$  to  $N_i \times \text{cone}(\Sigma_i)$  by collapsing  $N_i \times \text{spine}(W_i)$  onto  $N_i \times \text{point}$ . By iterating this process we get a resolution tower:

$$\tilde{M} = \tilde{M}_0 \xrightarrow{\pi_1} \tilde{M}_1 \xrightarrow{\pi_2} \dots \rightarrow \tilde{M}_{k-1} \xrightarrow{\pi_k} M$$



with  $\tilde{M}$  a smooth manifold. In fact by proving a generalized version of Proposition 4.2 we can adjust  $W_i$  so that each  $W_i$  has a spine  $S_i$  consisting of transversally intersecting  $A_{k-1}$  spaces without boundaries, and then each map  $\pi_k$  collapses  $N_i \times S_i$  to  $N_i \times \text{point}$ . This makes  $\pi: \tilde{M} \rightarrow M$ , where  $\pi = \pi_k \circ \dots \circ \pi_1$ , very much analogous to a multiblowup.

**THEOREM 5.1** ([AK<sub>6</sub>]). *The interior of any compact  $A$ -space is homeomorphic to a real algebraic set. Furthermore the natural stratification on this algebraic set coincides with the stratification of the  $A$ -structure.*

Theorem 5.3 tells that the class of  $A$ -spaces contain all compact P.L. manifolds hence:

**COROLLARY 5.2.** *The interior of any compact P.L. manifold is P.L. homeomorphic to a real algebraic set.*

The idea of the proof Theorem 5.1 goes as follows. First define  $\mathcal{O}_*(V)$ , a bordism group for an algebraic set  $V$ . It is the usual bordism group of maps of  $A$ -spaces into  $V$  modulo the subgroup generated by maps  $X \times N \rightarrow N \rightarrow V$  where  $X$  is an  $A$ -space,  $N$  is a nonsingular algebraic set and the map is the projection followed by an entire rational map  $N \rightarrow V$ . Then inductively we prove a generalized version of Theorem 2.8: that is if  $M \subset V$  is an imbedding of a compact  $A$ -space without boundary into a nonsingular algebraic set  $V$  such that  $M$  represents 0 in  $\mathcal{O}_*(V)$ , then  $M$  can be moved to an algebraic subset  $Z$  of  $V \times \mathbf{R}^n$  by a small isotopy (for some  $n$ ). This implies the proof of Theorem 5.1 (by taking  $V = \mathbf{R}^n$ ). Because one point compactification of an interior of a compact  $A$ -space is a compact  $A$ -space without boundary hence is homeomorphic to an algebraic set by above (and use Proposition 3.1 (b)).

Roughly the proof of the above claim proceeds as follows. Let  $M = M_0 \cup N \times \text{cone}(\Sigma) \subset V$  then the bordism condition on  $M$  implies that  $[N] \in \eta_*^4(V)$ , so by Theorem 2.8 we can assume that  $N$  is a nonsingular algebraic subset of  $V \times \mathbf{R}^m$  for some  $m$ . Define  $B_1(V \times \mathbf{R}^m, N) = B(V \times \mathbf{R}^m \times \mathbf{R}, N \times 0)$ , then this contains a natural nonsingular algebraic subset  $N_1(V \times \mathbf{R}^m, N) = B(N \times \mathbf{R}, N \times 0)$  which is diffeomorphic to  $N$ . By continuing in this fashion let

$$B_k(V \times \mathbf{R}^m, N) = B(B_{k-1}(V \times \mathbf{R}^m, N) \times \mathbf{R}, N_{k-1}(V \times \mathbf{R}^m, N) \times 0),$$

$$N_k(V \times \mathbf{R}^m, N) = B(N_{k-1}(V \times \mathbf{R}^m, N) \times \mathbf{R}, N_{k-1}(V \times \mathbf{R}^m, N) \times 0).$$

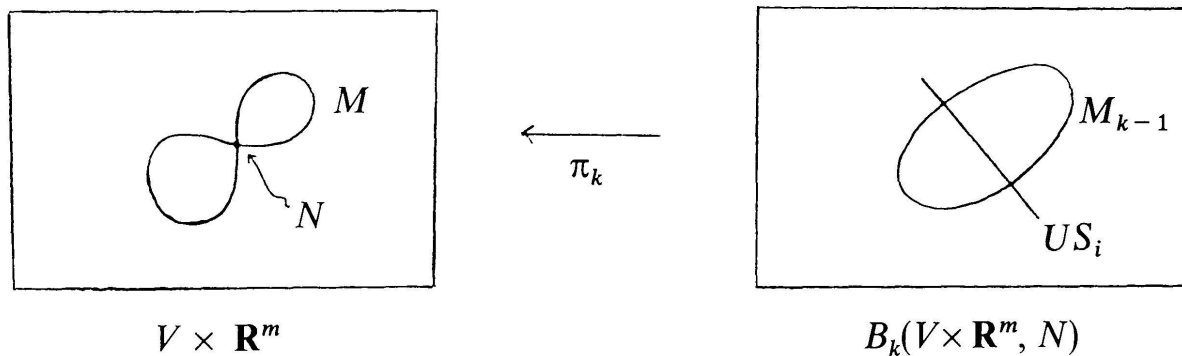
Then we get a generalized algebraic multiblowup  $\pi_k: B_k(V \times \mathbf{R}^m, N) \rightarrow V \times \mathbf{R}^m$

such that  $\pi_k^{-1}(N)$  is a union of codimension one submanifolds  $\bigcup S_i$  in general position and

$$\pi_k^{-1}(V \times \mathbf{R}^m - N) = (V \times \mathbf{R}^m - N) \times \mathbf{R}^k.$$

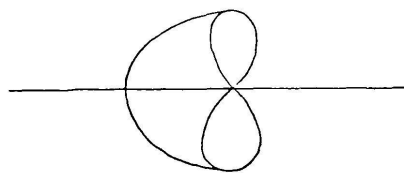
Since  $M$  is an  $A_k$ -space,  $\Sigma = \partial W$  for some compact  $A_{k-1}$ -space  $W$ . By proving a generalized version of Proposition 4.2 we can assume that the spine of  $W$  is a transversally intersecting codimension one  $A_{k-1}$  subspaces  $\bigcup L_i$  with  $\partial L_i = \emptyset$ . We then imbed the  $A_{k-1}$  space  $M_{k-1} = M_0 \cup N \times W$  (blown up  $M$ ) into  $B_k(V \times \mathbf{R}^m, N)$  such that

- (i)  $M_{k-1}$  is transversal to  $\bigcup S_i$  with  $M_{k-1} \cap \bigcup S_i = N \times \bigcup L_i$ ,
- (ii)  $\pi_k(M_{k-1})$  is isotopic to  $M$  by a small isotopy,
- (iii)  $M_{k-1}$  represents 0 in  $\mathcal{O}_*(B_k(V \times \mathbf{R}^m, N))$ .



This is somewhat hard to prove (see [AK<sub>6</sub>]). Then by induction, with a small isotopy  $M_{k-1}$  can be moved to an algebraic subset  $Z$  of  $B_k(V \times \mathbf{R}^m, N) \times \mathbf{R}^s$  for some  $s$ . Hence  $Z$  still satisfies (i) and (ii), after composing  $\pi_k$  with the obvious projection. Then by using a version of Proposition 3.3 we blow down  $Z$  to get an algebraic set homeomorphic to  $M$ .

The class of  $A$ -spaces does not contain all algebraic sets. For example the Whitney umbrella  $x^2 = zy^2$  is not an  $A$ -space.



Therefore to classify real algebraic sets we need a bigger class of resolvable spaces (§6).

In order to show that P.L. manifolds admit  $A$ -structures one has to appeal to algebraic topological methods. This is done in [AT<sub>2</sub>], here is a brief summary of

[AT<sub>2</sub>]: One first verifies that  $A_k$ -structures on P.L. manifolds obey the usual structure axioms ([L]). For example they satisfy the product structure axiom i.e. for any P.L. manifold  $M$  an  $A_k$ -structure  $(M \times I)_\gamma$  on  $M \times I$  is concordant to  $M_\gamma \times I$  where  $M_\gamma$  is an  $A_k$ -structure on  $M$ . Using [W] we can define an  $r$ -dimensional  $A_k$ -thickening on  $X$  to be a simple homotopy equivalence  $X \xrightarrow{f} W^r$  where  $W^r$  is an  $r$ -dimensional  $A_k$ -manifold (with boundary). Let  $T_k^r(X)$  to be the set of all  $r$ -dimensional  $A_k$ -thickenings on  $X$  with the equivalence relation:  $(W_1, f_1) \sim (W_2, f_2)$  if there is an  $(r+1)$ -dimensional  $A_k$ -thickening  $(W, F)$  with  $\partial W = W_1 \cup W_2$  and making the following diagram commute up to homotopy:

$$\begin{array}{ccc} & & W_1 \\ & \nearrow f_1 & \searrow \\ X & \xrightarrow{F} & W \\ & \searrow f_2 & \nearrow \\ & & W_2 \end{array}$$

There are natural maps  $T_k^r(X) \rightarrow T_k^{r+1}(X)$  given by  $(W, f) \mapsto (W \times I, f \times id)$ , so using these maps we can take the direct limit  $T_k(X) = \varinjlim T_k^r(X)$ . It follows that the functor  $X \mapsto T_k(X)$  is a representable functor (see [Sp]), hence by Brown representability theorem there exists a classifying space  $B_{A_k}$  such that  $T_k(X) = [X; B_{A_k}]$ . There are natural inductions  $B_{A_{k-1}} \rightarrow B_{A_k}$ , and let  $B_A = \varinjlim B_{A_k}$ . There is a natural forgetful map  $B_A \xrightarrow{\pi} B_{PL}$ . Then one shows that the usual structure theorem holds: Namely that a compact P.L. manifold  $M$  has an  $A$ -structure if and only if the normal bundle map (thickening map)  $M \xrightarrow{\nu_M} B_{PL}$  lifts to  $B_A$ . Let  $PL/A$  be the homotopy theoretical fibre of  $\pi$ , then:

THEOREM 5.3 ([AT<sub>2</sub>]).  $B_A \xrightarrow{\pi} B_{PL}$  is a trivial fibration, i.e.  $B_A \simeq B_{PL} \times PL/A$  and  $PL/A$  is a product of Eilenberg-McLain spaces  $K(\mathbb{Z}/2\mathbb{Z}, n)$ 's. The number  $\rho_n$  of  $K(\mathbb{Z}/2\mathbb{Z}, n)$  for each  $n$  in this product is given by

$$\rho_n = \begin{cases} 0 & \text{if } n < 8, \\ 26 & \text{if } n = 8, \\ \text{infinite but countable} & \text{if } n > 8. \end{cases}$$

COROLLARY 5.4. Every compact P.L. manifold  $M$  has an  $A$ -structure and the number of different  $A$ -structures (up to  $A$ -concordance) on  $M$  is given by

$$\bigoplus_{n \geq 8} H^n(M; \pi_n(PL/A)).$$

Briefly the proof of Theorem 5.3 goes as follows: By a standard argument,  $\pi_i(PL/A_k)$  coincides with the concordance classes of  $A_k$ -structures on  $S^i$  (the exotic  $A_k$ -spheres). Since  $\pi_i(PL/A) = \lim_{\leftarrow} \pi_i(PL/A_k)$  it follows by definitions that the inclusion  $\pi_i(PL/A) \rightarrow \eta_i^A$  is an injection, where  $\eta_i^A$  is the cobordism group of  $i$ -dimensional  $A$ -manifolds. Then we construct a Thom space  $MA$  such that  $\pi_i(MA) \approx \eta_i^A$  (by using a transversality argument for  $A$ -manifolds). Then it turns out that the map  $\eta_i^A \rightarrow H_i(B_A; \mathbf{Z}/2\mathbf{Z})$  given by  $\{M \xrightarrow{v_M} B_A\} \mapsto (v_M)_* [M]$  is an injection. We can put these maps into the following commutative diagram:

$$\begin{array}{ccccc} & \pi_i(PL/A) & \rightarrow & \eta_i^A & \\ & \downarrow f & & \downarrow & \\ h \swarrow & & & & \\ H_i(PL/A; \mathbf{Z}) & \xrightarrow{r} & H_i(PL/A; \mathbf{Z}/2\mathbf{Z}) & \xrightarrow{g} & H_i(B_A; \mathbf{Z}/2\mathbf{Z}) \end{array}$$

where  $h$  is the Hurewicz map,  $r$  is the reduction and  $g$  is induced by inclusion. Since the other two maps are injections then  $f$  must be injection. In fact  $f$  is a split injection since it is a map between  $\mathbf{Z}/2\mathbf{Z}$ -vector spaces. Hence  $h$  is a split injection. This implies that all  $k$ -invariants of  $PL/A$  is zero, i.e.  $PL/A$  is a product of Eilenberg-McClaine spaces  $\prod K(\mathbf{Z}/2\mathbf{Z}, n_i)$ . Then by dualizing the split injection  $g \circ f$  we get a surjection

$$H^i(B_A; \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\lambda} \text{Hom}(\pi_i(PL/A); \mathbf{Z}/2\mathbf{Z})$$

Let  $\delta_{n_i} \in H^{n_i}(B_A; \mathbf{Z}/2\mathbf{Z})$  such that  $\lambda(\delta_{n_i})$  is the generator of  $\mathbf{Z}/2\mathbf{Z}$ .

$$\delta = \prod \delta_{n_i} \text{ defines a map } B_A \rightarrow \prod_i K(\mathbf{Z}/2\mathbf{Z}, n_i) = PL/A.$$

Then the map  $\pi \times \delta: B_A \rightarrow B_{PL} \times PL/A$  turns out to be the desired splitting. The calculation of  $\rho_n$  can be done by using the geometric interpretation of  $\pi_*(PL/A)$ .

The set  $\mathcal{S}_A(M) = \bigoplus_n H^n(M; \pi_n(PL/A))$  measures the number of different "topological resolutions" of  $M$ , up to concordance (i.e.  $A$ -structures). Therefore often  $\mathcal{S}_A(M)$  is infinite; and  $\mathcal{S}_A(M^8)$  has  $2^{2^6}$  elements for any closed 8-manifold  $M^8$ .

## §6. ON CLASSIFICATION OF REAL ALGEBRAIC SETS

The resolution and complexification properties of real algebraic sets impose many restrictions on the underlying topological spaces. To give a topological characterization of algebraic sets one has to find all such properties, such that a