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PROPOSITION 3.3. Given  $L, W, V$  as above, then  $D_q(V) \cup L$  is an algebraic subset of  $W \times \mathbf{R}^n$ .

*Proof:* Let  $p: \mathbf{R}^m \times \mathbf{R}^n \rightarrow \mathbf{R}$  be an overt polynomial of degree  $e$  with  $V = p^{-1}(0)$  and let  $q$  be as above. Define a polynomial  $r: \mathbf{R}^m \times \mathbf{R}^n \rightarrow \mathbf{R}$  by

$$r(x, y) = q(x)^e p\left(x, \frac{y}{q(x)}\right)$$

We claim  $r^{-1}(0) = D_q(V) \cup L$ . It is easy to see that

$$r^{-1}(0) \cap (W - L) \times \mathbf{R}^n = D_q(V) \cap (W - L) \times \mathbf{R}^n,$$

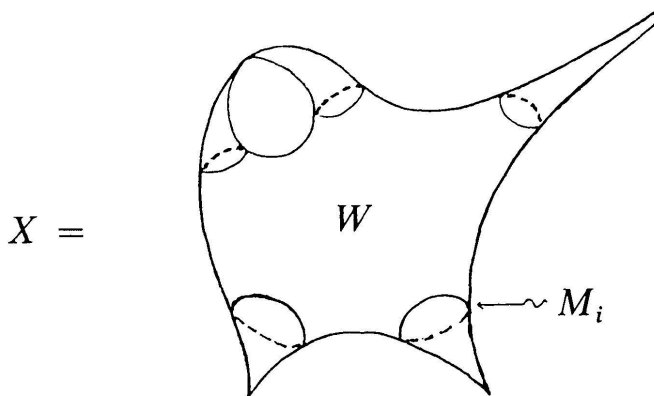
so it suffices to show that  $r^{-1}(0) \cap (L \times \mathbf{R}^n) = L \times 0$ . We decompose  $p(x, y) = p_e(x, y) + \alpha(x, y)$  where  $p_e(x, y)$  is homogeneous of degree  $e$  and  $\alpha(x, y)$  is a polynomial of degree less than  $e$ . Hence if  $(x, y) \in r^{-1}(0) \cap (L \times \mathbf{R}^n)$  then  $r(x, y) = 0$  and  $q(x) = 0$ , which implies  $r(x, y) = p_e(0, y) = 0$ . Then  $y = 0$  since  $p$  is overt, so  $(x, y) \in L \times 0$ . Conversely if  $(x, y) \in L \times 0$  then  $y = 0$  and  $q(x) = 0$ . Hence  $r(x, y) = p_e(0, 0) = 0$ , i.e.  $(x, y) \in r^{-1}(0) \cap (L \times \mathbf{R}^n)$ .  $\square$

There is a more useful version of Proposition 3.3 which says that after modifying  $D_q$  we can get  $D_q(V) \cup L$  to be a projectively closed algebraic set (Proposition 3.1 of [AK<sub>6</sub>]). This allows us to iterate this blowing down process.

#### §4. ISOLATED SINGULARITIES

The topology of real algebraic sets with isolated singularities is completely understood by the following Theorem.

THEOREM 4.1 ([AK<sub>2</sub>]).  $X$  is homeomorphic to an algebraic set with isolated singularities if and only if  $X$  is obtained by taking a smooth compact manifold  $W$  with boundary  $\partial W = \bigcup_{i=1}^r M_i$ , where each  $M_i$  bounds, then crushing some  $M_i$ 's to points and deleting the remaining  $M_i$ 's.

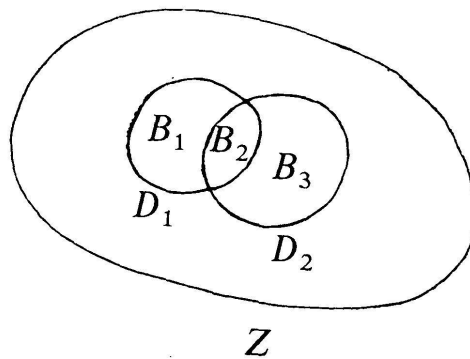


One direction the proof follows from the resolution of singularities [H]. To prove it to the other direction we need the following:

PROPOSITION 4.2. *If a closed smooth manifold  $M$  bounds a compact manifold, then it bounds a compact manifold  $W$  such that there are transversally intersecting closed smooth codimension one submanifolds  $W_1, \dots, W_r$  with  $W/\cup W_i \approx \text{con}(M)$ , in other words  $\cup W_i$  is a spine of  $W$ .*

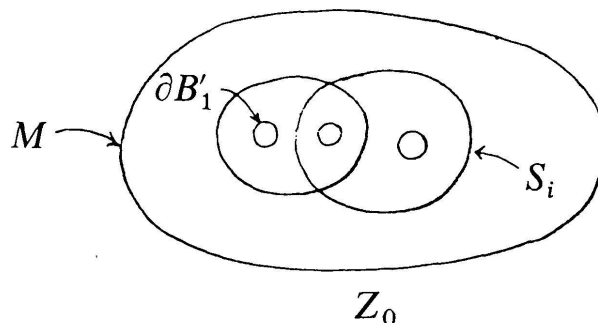
*Proof:* Let  $M = \partial Z$  where  $Z$  is some closed smooth manifold. Then pick balls  $D_i, i = 1, 2, \dots, r$  lying in interior ( $Z$ ) such that:

- (a)  $\cup_i D_i$  is a spine of  $Z$
- (b) The spheres  $S_i = \partial D_i$  intersect transversally with each other in  $Z$
- (c)  $\cup D_i - \cup \partial D_i$  is a union of open balls  $\cup_{j=1}^s B_j$ .

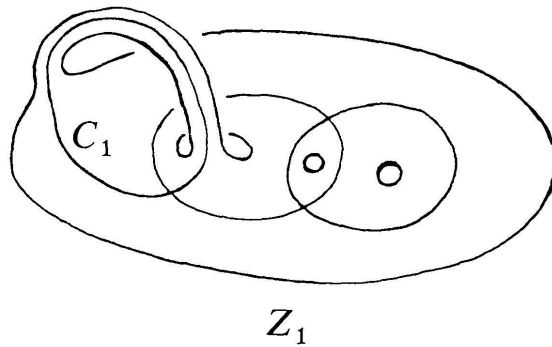


Let  $B'_j \subset B_j$  denote a smaller ball. Then  $Z_0 = Z - \bigcup_{j=1}^s \text{interior}(B'_j)$  is a manifold with spine  $\cup S_i$ , and

$$\partial Z_0 = M \cup \bigcup_{j=1}^s \partial B'_j, \quad \partial B'_j \approx S^m$$

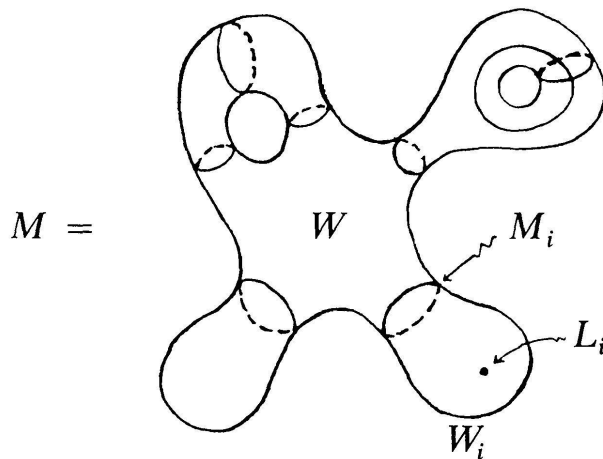


Order  $\{B'_j\}$  so that there is an arc from  $M$  to  $\partial B'_1$  intersecting exactly one  $S_i$ . Then attach a 1-handle to  $\partial Z_0$  connecting  $M$  to  $\partial B'_1$  get  $Z_1 = Z_0 \cup (1\text{-handle})$  as in the figure:



Then  $\partial Z_1 = M \cup \bigcup_{j=2}^s \partial B'_j$  and  $\bigcup S_i \cup C_1$  is a spine of  $Z_1$ , where  $C_1$  is the circle defined by the core of the 1-handle union of the arc. By continuing this fashion we get  $Z_s$  with  $\partial Z_s = M$ ; and the spine of  $Z_s$  is transversally intersecting codimension one spheres and circles  $\bigcup S_i \cup \bigcup_{j=1}^s C_j$ . We are finished except  $C_j$  are not codimension one. We remedy this by topologically blowing up  $Z_s$  along  $\bigcup C_j$ , i.e. let  $W = B(Z_s, \bigcup C_j)$  and let  $W_i$  to be the projectified normal bundles  $P(C_j, Z_s)$  of  $C_j$  (i.e. the blown up circles), and  $B(S_i, S_i \cap \bigcup C_j)$  we are done.  $\square$

*Proof of Theorem 4.1:* By Proposition 3.1 it suffices to prove this for one point compactification of  $X$ . Hence we can assume that  $X$  is compact. Let  $W$  be a compact smooth manifold,  $\partial W = \bigcup_{i=1}^r M_i$  and each  $M_i$  bounds. By Proposition 4.2 we can assume  $M_i = \partial W_i$  such that each  $W_i$  has a spine consisting of union of transversally intersecting codimension one closed smooth submanifolds  $L_i$ . Let  $M = W \cup_{\partial} \bigcup W_i$



By Theorem 2.12 we can assume that the manifolds  $(M; L_1, \dots, L_r)$  are pairwise diffeomorphic to nonsingular algebraic sets  $(Z; Z_1, \dots, Z_r)$ . Let  $h : Z \rightarrow \mathbf{R}$  be an entire rational function with  $h|_{Z_i} = i$  ( $h$  exists by Lemma 0.1). Let  $\lambda : Z \rightarrow \mathbf{R}$  be a polynomial with  $\lambda^{-1}(0) = \cup_i Z_i$ . By Proposition 3.2 there exists a nonsingular projectively closed algebraic set  $V \subset \mathbf{R}^2 \times \mathbf{R}^n$  and a birational diffeomorphism  $g$  making the following commute

$$\begin{array}{ccc} V & \hookrightarrow & \mathbf{R}^2 \times \mathbf{R}^n \\ g \uparrow \approx & & \downarrow \pi \\ Z & \xrightarrow{f} & \mathbf{R}^2 \end{array}$$

where  $f = (h, \lambda)$ . Let  $L = \{(1, 0), (2, 0), \dots, (r, 0)\}$  then by Proposition 3.3 we can blow down  $V$  over  $L$  algebraically. This gives an algebraic set homeomorphic to  $X$ . □

**COROLLARY 4.3.** *Up to diffeomorphism nonsingular algebraic sets are exactly the interiors of compact smooth manifolds with boundary (possibly empty).*

The following is a local knottedness theorem of real algebraic sets. It is an ambient version of Theorem 4.1. It says that unlike complex algebraic sets all knots can occur as links of singularities.

**THEOREM 4.4** ([AK<sub>4</sub>]). *Let  $W^m$  be a compact smooth submanifold of  $S^{n-1}$  imbedded with trivial normal bundle with codimension  $\geq 1$ . Then there exists an algebraic set  $V \subset \mathbf{R}^n$  with  $\text{Sing}(V) = \{0\}$  such that  $(B_\varepsilon, B_\varepsilon \cap V) \approx (B^n, \text{cone}(\partial W))$  for all small  $\varepsilon > 0$ , where  $B_\varepsilon$  is the ball of radius  $\varepsilon$  centered at 0. In fact  $\varepsilon(\partial W)$  is isotopic to  $\partial B_\varepsilon \cap V$  in  $\partial B_\varepsilon$ .*

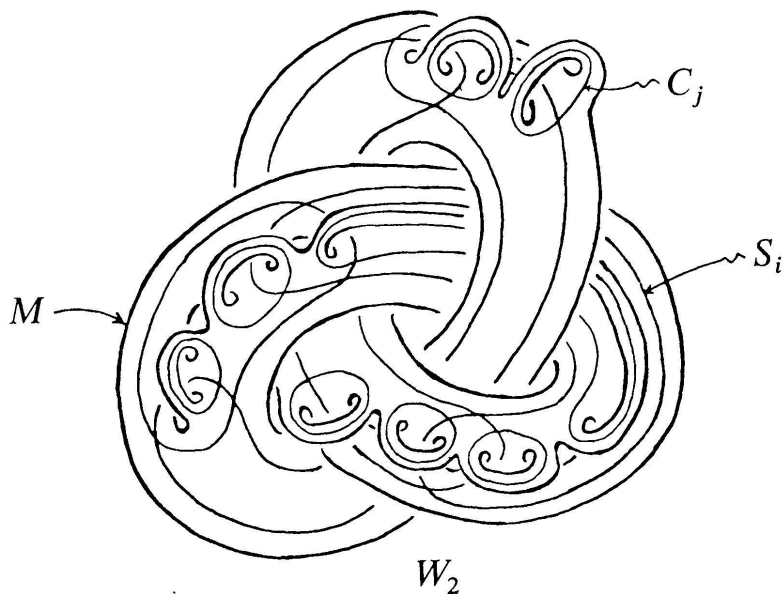
By taking  $W$  to be the Seifert surface of a knot we get an interesting fact.

**COROLLARY 4.5.** *Any knot  $K^{n-3} \subset S^{n-1}$  is isotopic to a link of an algebraic set  $V$  in  $\mathbf{R}^n$ .*

*A sketch proof of Theorem 4.4:* First identify  $W \subset \mathbf{R}^{n-1} \approx S^{n-1} - \infty$ , and call  $M = \partial W$ . Then apply the process of getting nice spines to  $W^m$  (Proposition 4.2); i.e. pick a family of discs  $D_i, i = 1, \dots, r$  in  $W$  whose boundaries are in general position, and  $W/\cup D_i \approx \text{cone}(M)$  and  $\cup D_i - \cup S_i$  is a disjoint union of open balls  $\cup B_j$  where  $S_i = \partial D_i$ . Let  $W_1$  be the manifold obtained by removing a small open ball from each  $B_j$ . Now by attaching 1-handles to  $W_1$  as in

Proposition 4.2 we obtain  $W_2$ , whose spine consists of  $\bigcup S_i$  union circles  $\bigcup C_j$ , with  $\partial W_2 = M$ .

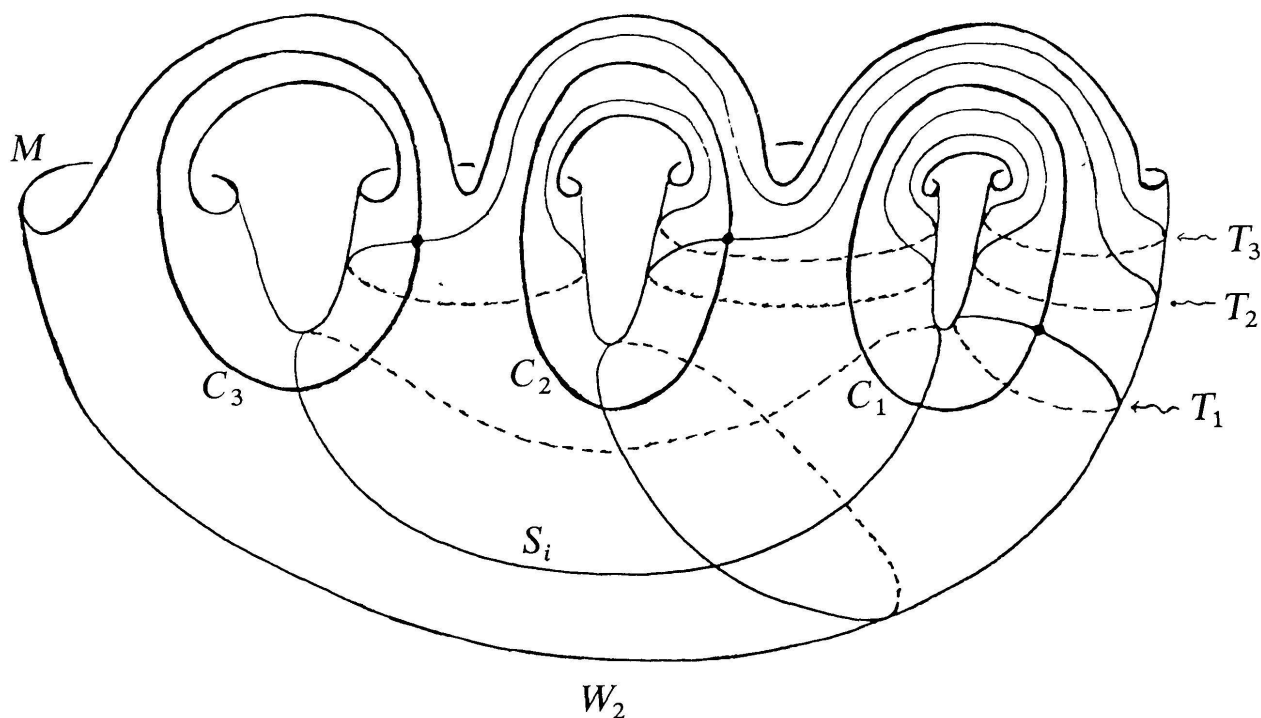
Observe that this whole process can be done inside  $\mathbf{R}^{n-1}$  and  $C_j$  and  $S_i$  are unknotted in  $\mathbf{R}^{n-1}$



We claim that there is disjointly imbedded  $m - 1$  spheres  $T_j, j = 1, \dots, s$  in  $W_2$  such that

- (1) Each  $T_j$  is unknotted in  $\mathbf{R}^{n-1}$ .
- (2) Each  $T_j$  meets  $C_j$  at a single point, and  $T_j \cap C_i = \emptyset$  for  $i \neq j$ .
- (3) For each  $i$  there is  $B_i \subset \{1, 2, \dots, s\}$  so that  $S_i \cup \bigcup_{j \in B_i} T_j$  separates  $W_2$ .

This can be easily done as in the following picture.



(1) and (2) are easily checked from the picture. To see (3), let  $B_i = \{j \mid C_j \cap S_i \neq \emptyset\}$ .

Let  $W_3 = W_2 \cup_{\partial} -W_2$ . The imbedding  $W_2 \subset \mathbf{R}^{n-1}$  can be extended to an imbedding of  $W_3$ . Since  $T_j$  and  $C_j$  are unknotted and by (2), we can isotop  $W_3$  so that  $T_j \cup C_j$  in  $W_3$  coincides with  $S^{m-1} \cup S^1$  in  $(S^{m-1} \times S^1)_j$ , where  $(S^{m-1} \times S^1)_j, j = 1, \dots, s$  are disjointly imbedded copies of the standard  $S^{m-1} \times S^1$  in  $\mathbf{R}^{n-1}$ . We can assume that some open neighborhoods of these sets in  $W_3$  and  $(S^{m-1} \times S^1)_j$  also coincide. By Theorem 2.3 and Remark 2.4 we can isotop  $W_3$  to a component of a nonsingular algebraic set  $Z$  fixing  $T_j \cup C_j$  for all  $j$ . In fact after a minor adjustment (to proof of Theorem 2.3) we can assume that  $Z$  is projectively closed. Continue to call isotoped copy of  $S_i$  by  $S_i$ .

Since as codimension one homology classes  $[S_i] = [\bigcup_{j \in B_i} T_j]$  and  $\bigcup_{j \in B_i} T_j$  is a nonsingular algebraic set,  $S_i$  can be made a nonsingular algebraic set for each  $i$  (Theorem 2.6). Hence the spine  $L = \bigcup S_i \cup \bigcup C_j$  of  $W_2 \subset Z$  can be assumed to be an algebraic set. Since  $Z$  is projectively closed so is  $L$ .

Let  $p, q$  be overt polynomials with  $p^{-1}(0) = Z$  and  $q^{-1}(0) = L$ . Define

$$V = \{(x, t) \in \mathbf{R}^{n-1} \times \mathbf{R} \mid t^{2e+1} = q^*(x, t)^2, p^*(x, t) = 0\}$$

where  $p^*(x, t) = t^d p(x/t), q^*(x, t) = t^e q(x/t)$  where  $d = \text{degree } p, e = \text{degree } q$ . If  $(x, t) \in V$  then  $t \geq 0$ ; and if  $t = 0$  then  $x = 0$  since  $p$  is overt.

$$(\mathbf{R}^{n-1} \times \varepsilon, (\mathbf{R}^{n-1} \times \varepsilon) \cap V) \approx (\mathbf{R}^{n-1}, q^{-1}(\varepsilon) \cap Z) \approx (\mathbf{R}^{n-1}, M),$$

since  $q^{-1}(\varepsilon) \cap Z \approx \partial W_2 = M$ . We are almost done.

Let  $S_\varepsilon^{n-1} = \{(x, t) \in \mathbf{R}^{n-1} \times \mathbf{R} \mid |x|^2 + t^2 = \varepsilon^2\}$ , and  $\varphi_\varepsilon: \mathbf{R}^{n-1} \rightarrow S_\varepsilon^{n-1}$  be the imbedding  $\varphi_\varepsilon(y) = (1 + |y|^2)^{-1/2}(\varepsilon y, \varepsilon)$ . Then

$$\varphi_\varepsilon^{-1}(S_\varepsilon^{n-1} \cap V) = \{y \in \mathbf{R}^{n-1} \mid p(y) = 0, q^4(y)(1 + |y|^2) = \varepsilon^2\}$$

which is isotopic to  $M$  in  $\mathbf{R}^{n-1}$  for all small  $\varepsilon > 0$ . Hence  $(S_\varepsilon^{n-1}, S_\varepsilon^{n-1} \cap V) \approx (S_\varepsilon^{n-1}, M)$  for all small  $\varepsilon > 0$ . □

