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SOME PARADOXICAL SETS WITH APPLICATIONS IN THE GEOMETRIC THEORY OF REAL VARIABLE [1])

by Miguel de GUZMÁN

The purpose of this paper is to present a small excursion through a certain area of the theory of real variable, describing some strange constructions, paradoxical and beautiful in their own way, that have recently come to illuminate some other important topics in related fields such as Fourier analysis. We shall do it in an expository way, trying to avoid most of the technicalities. For them we refer the reader to the works of the author published in 1975 and 1981.

1. MANEUVERING A NEEDLE

In 1917 Kakeya proposed a curious problem with the aspect of a puzzle. It can be formulated in the following way. Let us consider a one-dimensional car, like a straight needle of one meter of length, located on the plane. We can maneuver the needle on its plane in a continuous way until placing it in the same plane it occupies but in inverted position. In doing so the needle will sweep a certain area. The question is: *What is the minimal value of the areas of the figures within which the needle can be continuously inverted?*

A circle of radius $1/2$ (area: $\pi/4 = 0.785398...$) with center at the middle point of the needle is such a figure in which the needle can rotate (Fig. 1).

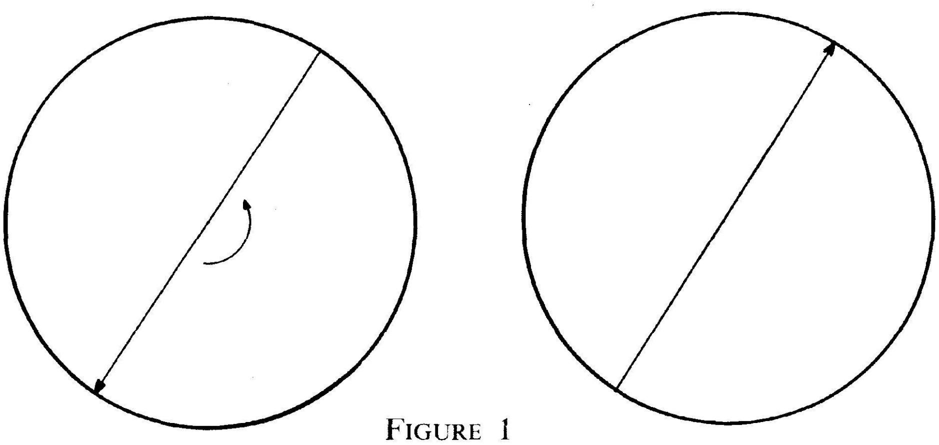

FIGURE 1

[1]) The present paper is an expanded version of a talk given at the Mathematics Department of the Universidade Federal do Rio de Janeiro.

But also the triangle of Figure 2 (area: $1/\sqrt{3} = 0.5773502... < \pi/4$) is another such figure of smaller area.

For a long time many people thought that the problem would be solved by means of a sort of curvilinear triangle (see Fig. 3), a figure bounded by a hypocycloid γ with three cuspidal points inscribed in a circle of radius 3/4. This curve has the property that for each point M in γ the tangent at M to γ intersects γ in two other points A and B such that the length of AB is 1. The area of the figure enclosed by it is $\pi/8 = 0.392699...$ and it is easy to see that a needle of length 1 can turn around in any open set containing such a figure inside.

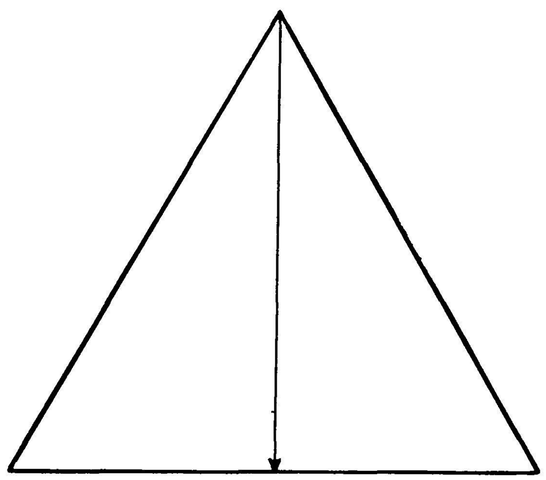

FIGURE 2

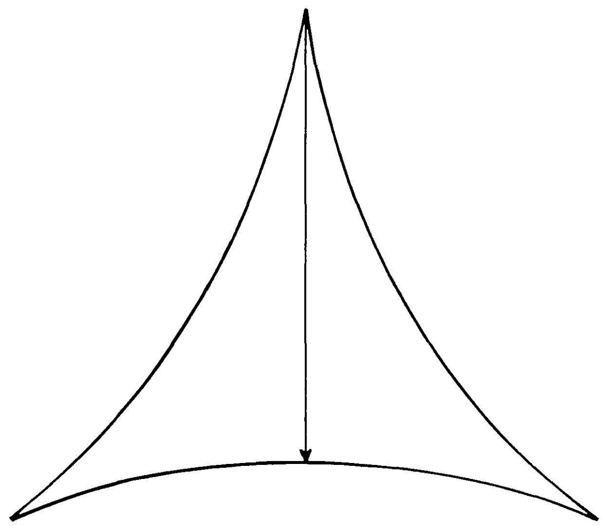

FIGURE 3

2. STREETS IN ALL DIRECTIONS COVERING NULL AREA

Kakeya's problem was in fact solved in 1919 by Besicovitch, but nobody, not even Besicovitch himself, realized it. He was at that time in Perm, in a University rather isolated from the rest of the mathematical world where the Kakeya problem did not arrive. On the other hand the tool created by Besicovitch for the solution of some other problem did not go very far from his place either. Besicovitch constructed a plane set of null area containing segments of length 1 in all directions. As we shall see this gives the following solution to the Kakeya problem: *Given any arbitrarily small* $\eta > 0$ *one can construct a plane figure with area smaller than* η *such that the needle can be continuously inverted inside it.*

3. A TOURIST COLONY NOT TO BE RECOMMENDED

In 1927 Nikodym, in order to explore the geometric structure of the measurable sets in the plane, showed how to construct, inside a square Q, a set N that fills it (i.e. the measure of Q-N is zero) and so that for each point x of N there is a straight line $l(x)$ passing through it and not hitting any other point of N (Fig. 4).

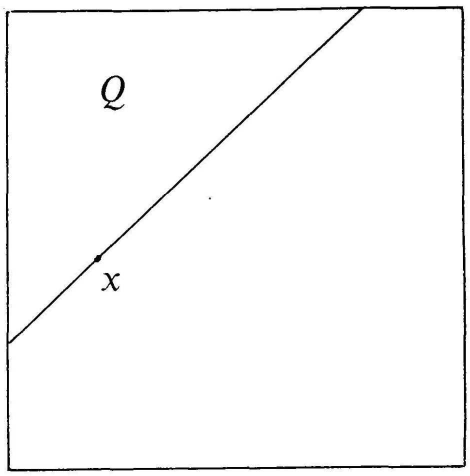

FIGURE 4

The architects of tourist colonies have not yet learned about this magnificent business possibility, but the day somebody tells you about the marvels of a colony in an island which offers a free view over the ocean from each one of its apartments, beware!

Although it seems incredible one can still make it better. R. O. Davies in 1953 constructed a set N in Q filling Q and such that each x of N has infinitely many directions in which one can see the ocean... inside any arbitrarily small angle one may fix!

4. A SMALL TREE WITH MANY FRUITS

In 1928 Besicovitch was informed about the needle problem and published its solution. In 1929 Perron simplified the somewhat laborious construction of Besicovitch. It has been further simplified later on. The final product of the line of thought, that we shall call *the Perron tree*, has proved to be an extraordinarily fruitful tool for the solution of certain deep problems of recent mathematical analysis.

The result is as follows: Given an arbitrary $\varepsilon > 0$ and an arbitrary triangle ABC of area that we denote by $S(ABC)$, we can divide the triangle ABC into small triangles $T_1, T_2, \ldots, T_n$ as Figure 5 shows (i.e. dividing the basis a into a finite number of equal intervals $I_1, I_2, \ldots, I_n$) and one can translate appropriately the small triangles $T_1, T_2, \ldots, T_n$ parallelly to the basis a in such a way that the area of the union of the translated triangles is less than $\varepsilon S(ABC)$. (See Fig. 6.)

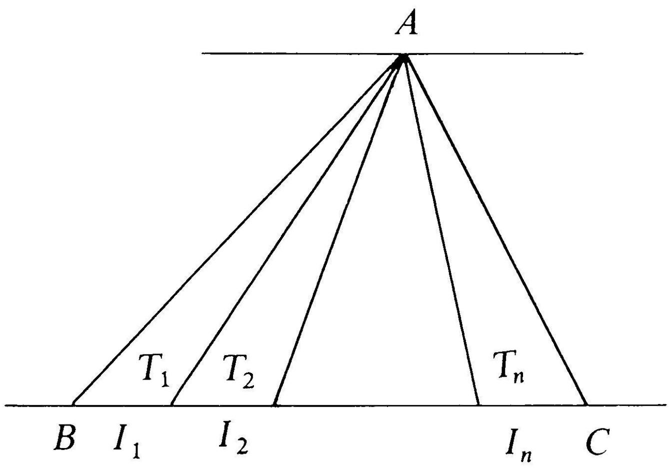

FIGURE 5

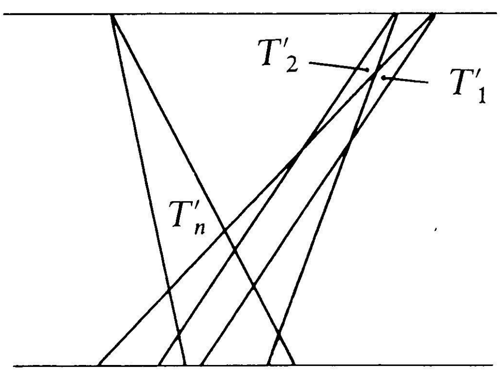

FIGURE 6

5. How the Perron tree sprouts

Following an idea of Rademacher (1962), the construction of the Perron tree can be easily understood as follows. Let us divide first a triangle T, MNP, of area $S(T)$, into two triangles T_1, T_2, with bases J_1, J_2, of the same length. If we wish to move T_1 and T_2, parallelly to NP so that the shifted triangles cover less area we can do it by pushing T_2 towards T_1 as Figure 7 shows. The area covered by T_1 and T'_2 can be easily measured by elementary geometry and is (see Fig. 7, we take $1/2 < \alpha < 1$)

$$\alpha^2 S(T) + 2(1-\alpha)^2 S(T)$$

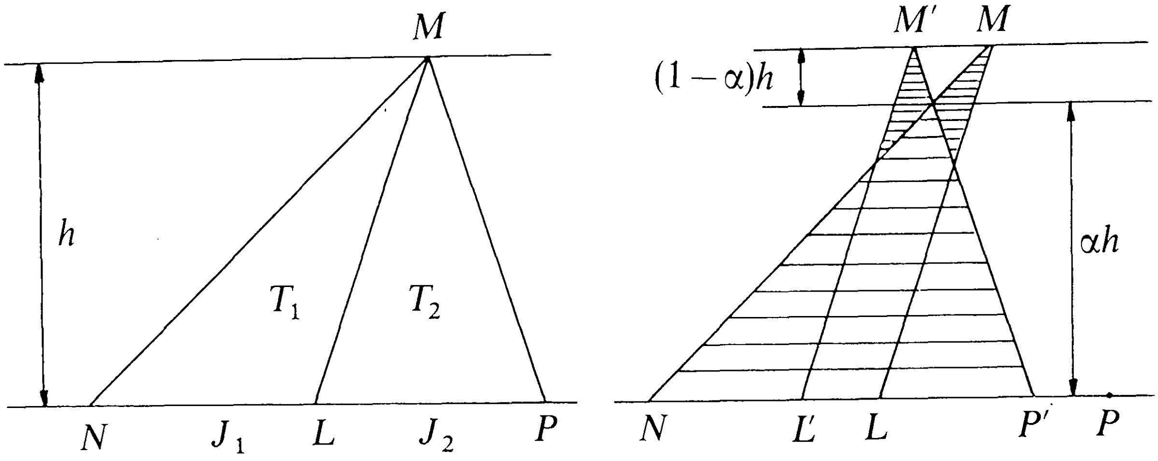

FIGURE 7

If the triangle MNP is divided into four parts, instead of two, as Figure 8 indicates, we can first subject the pair of triangles MNL_1 and ML_1L_2 on the one hand to the above indicated operation with an α, $1/2 < \alpha < 1$, and, on the other hand we can do the same, with the same α, to the other pair of adjacent triangles ML_2L_3, ML_3P. The result is indicated in Figure 8.

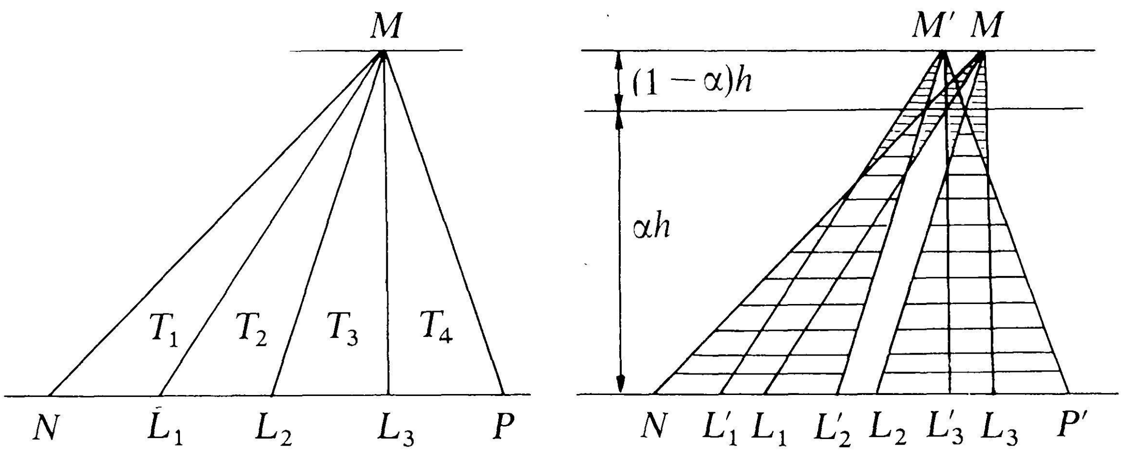

FIGURE 8

It is easy to see that the area of the figure now covered by the so translated triangles is less than

$$(*) \qquad \alpha^2 S(T) + 2(1-\alpha)^2 S(T)$$

If we now shift in a solidary way the figure formed by the union of the two triangles T_3 and T_4' towards the left until L_2 coincides with L_2', the new formed figure covered by the four triangles can be considered (see Fig. 9) as consisting of a triangle HNP'' similar to the first one MNP with a similarity ratio α plus four peak triangles that overlap more than before. The area of this figure is therefore less than (*).

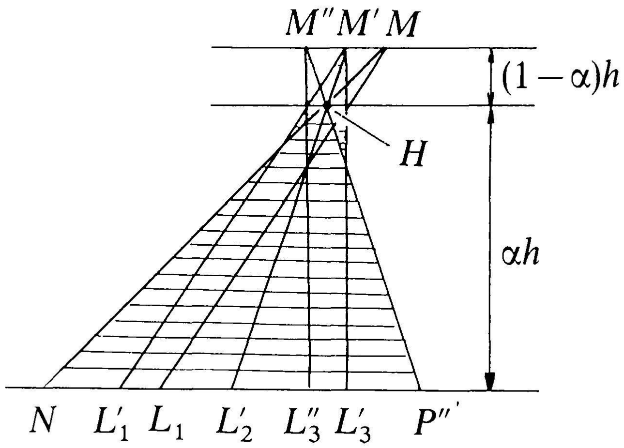

FIGURE 9

In the triangle HNP'' we have the basis divided into equal portions NL'_2 and L'_2P'' and so we can submit HNP'' to the initial operation, i.e. shifting the right hand triangle towards the left one with the same constant α that measures the magnitude of this shift and shifting thereby solidarily the triangles T''_3 and T''_4 that constitute the right hand portion of the triangle HNP''. The result is shown in Figure 10.

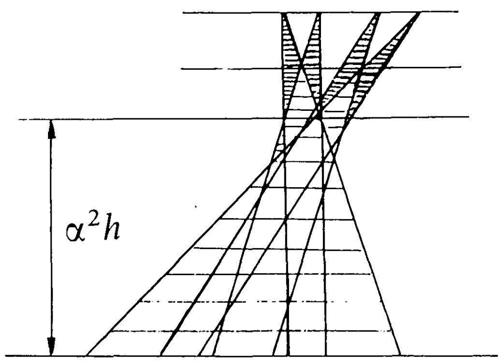

FIGURE 10

The final result is a triangle similar to the initial one with similarity ration α^2, its area therefore being $\alpha^4 S(T)$, plus four peaks that cover an area smaller than

$$2(1-\alpha)^2\alpha^2 S(T) + 2(1-\alpha)^2 S(T)$$

i.e. the total area now covered is not greater than

$$\alpha^4 S(T) + 2(1-\alpha)^2 (1+\alpha^2) S(T)$$

It is now not difficult to realize that if we initiate our process with 2^n equal portions of the basis and we proceed in a similar way, at the end, i.e. after n repetitions of the process consisting of (a) a shift of the right triangle of each pair of adjacent triangles towards the left triangle (shift constant $= \alpha$), and (b) gluing together the resulting figures to compose a triangle similar to the original one with one half the number of divisions on the basis, we obtain a figure with an area not greater than

$$\begin{aligned} &\alpha^{2n} S(T) + 2(1-\alpha)^2 (1+\alpha^2+\ldots+\alpha^{2n-2}) S(T) \\ &\leqslant \alpha^{2n} S(T) + 2(1-\alpha)^2 (1+\alpha^2+\ldots+\alpha^{2n-2}+\ldots) S(T) \\ &= \left(\alpha^{2n} + \frac{2(1-\alpha)^2}{1-\alpha^2}\right) S(T) \leqslant (\alpha^{2n}+2(1-\alpha)) S(T) \end{aligned}$$

Therefore, given $\varepsilon > 0$, we first choose α, $1/2 < \alpha < 1$ such that $1 - \alpha < \varepsilon/2$ and then n so that $\alpha^{2n} < \varepsilon/2$. In this way we obtain a Perron tree. Its name is justified by the fact that the final figure consists of a trunk (a triangle similar to the initial one with similarity ration α^n) plus many sharp branches that seem to rest on it. Its area is less than $\varepsilon S(T)$. (See Fig. 11.)

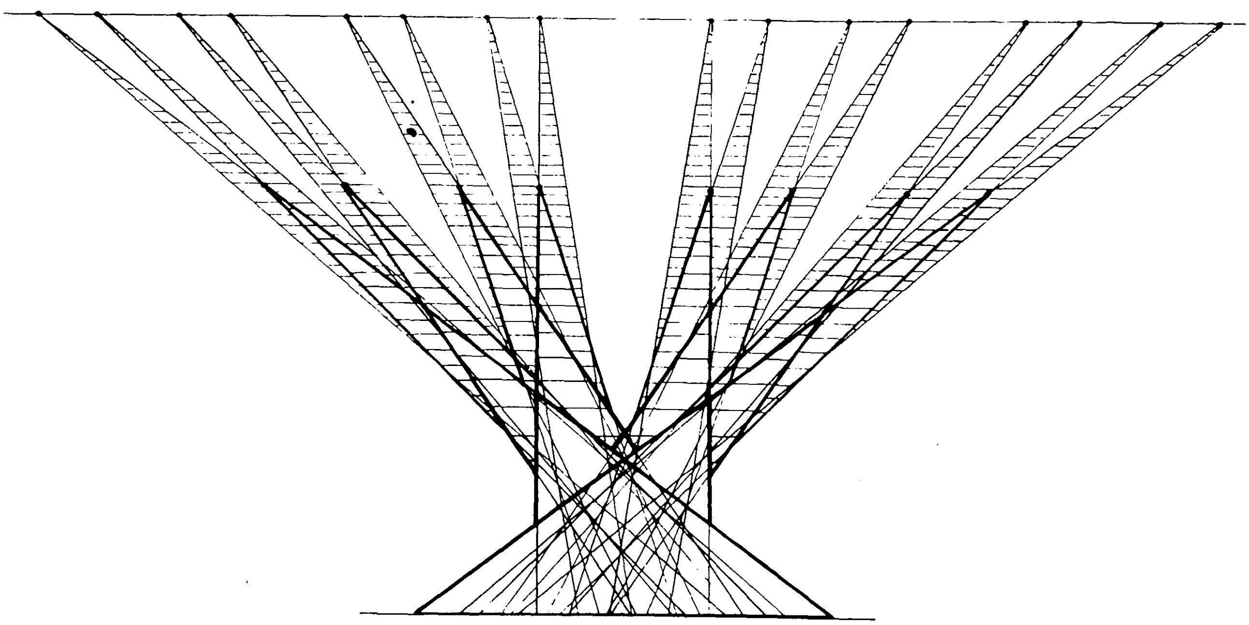

FIGURE 11

The reader interested in more details can consult Guzmán (1975).

6. THE SOLUTION OF THE NEEDLE PROBLEM

The Perron tree gives a simple solution to the Kakeya problem. First we shall show how a needle can go from a straight line to another one parallel to it covering an arbitrarily small area. Let us observe Figure 12.

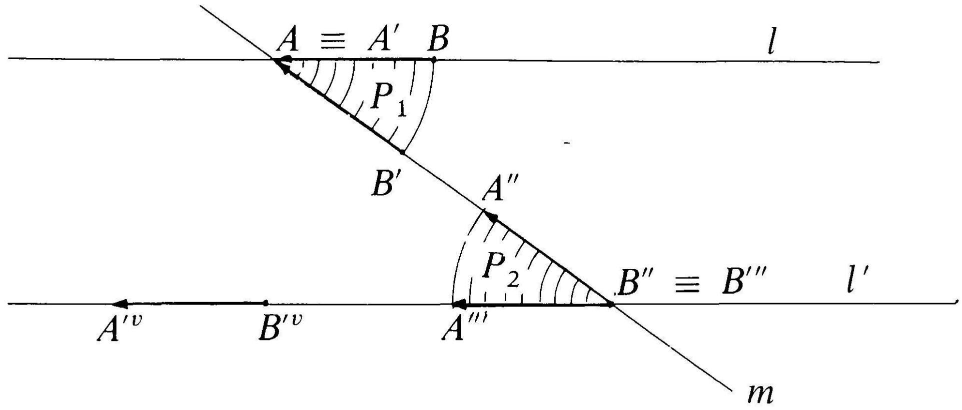

FIGURE 12

If the needle AB is on l and we wish to translate it to l^I, we draw through A a straight line m intersecting l and l^I whose direction can be as close to that of l and l^I as we wish. From AB we move to A^IB^I covering area P_1, from A^IB^I to $A^{II}B^{II}$ covering null area, from $A^{II}B^{II}$ to $A^{III}B^{III}$ covering P_2. Now $P_1 + P_2$ can be made arbitrarily small if the slope of m over l and l^I is small. From $A^{III}B^{III}$ we can move to any other position $A^{IV}B^{IV}$ on l^I covering again null area.

Let us now assume that the needle is on the side AB of the initial triangle ABC. We can assume that ABC is an equilateral triangle and that its height is of the same length as that of the needle. Let us see how we can move the needle to AC sweeping an area smaller than $\eta/3$ with a positive η arbitrarily small.

We construct a Perron tree P starting from ABC with an $\varepsilon > 0$ such that $\varepsilon S(ABC) < \eta/6$. Here, as before, $S(ABC)$ denotes the area of the triangle ABC. Let n be the number of small triangles $T_1, T_2, ..., T_n$ in which we have to divide ABC and let $T'_1 \equiv T_1, T'_2, ..., T'_n$ be their corresponding final positions in the Perron tree. We shall move the needle inside P and inside n figures like that of Figure 12 with an area J each one such that $nJ < \eta/6$. If the needle is on AB with an extremity on A, it can move inside $T'_1 \equiv T_1$, therefore inside P, until it comes over the right hand side of T'_1. Now T'_2 has its left hand side parallel to the right hand side of T'_1. Therefore it can move, using the above construction, covering an area J. Within T'_2, and so within P, it can move to the right hand side of T'_2. From there to the left hand side of T'_3 and so on until it comes to AC, covering area less than $\eta/3$.

It is clear that with three equilateral triangles and three repetitions of this process we can turn the needle around covering area smaller than η.

7. The construction of the Besocovitch set

The Besicovitch set is also easily built starting from the Perron tree by means of the following auxiliary construction:

(**) *Given an arbitrary parallelogram* $ABCD$ *and* $\varepsilon > 0$, *it is possible to construct a finite number of closed parallelograms* $\omega_1, \omega_2, \ldots, \omega_n$ *so that* (see Fig. 13):

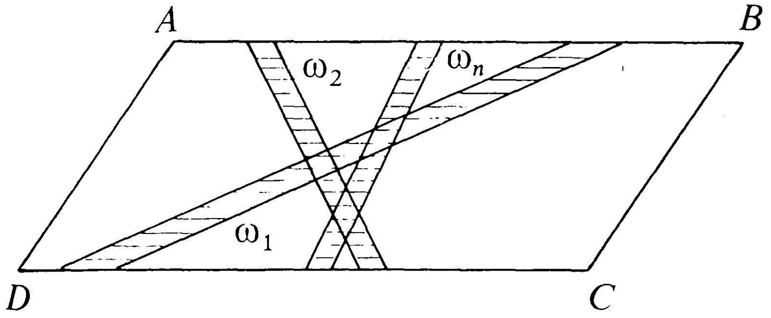

FIGURE 13

(a) *Each one has one basis on* AB *and another one on* CD.

(b) *The area of their union is less than* ε.

(c) *For each segment joining a point of* AB *to another one of* CD *there exists inside some* ω_j *a segment parallel to it of the same length.*

To see this, given $ABCD$ and $\varepsilon > 0$ we first take two strips ω_1 and ω_2 as indicated in Figure 14 such that $S(\omega_1) + S(\omega_2) < \varepsilon/4$. We take now a point L of UV so that LC is parallel to UT. Then we divide VC into intervals with the same length smaller than that of DV and we join L to the extreme points of these intervals. A typical triangle of the ones so obtained is LMN. Let p be

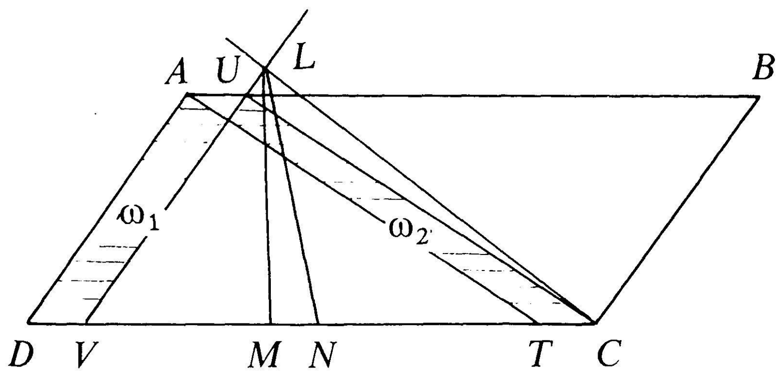

FIGURE 14

the number of such triangles. From LMN we construct a Perron tree with area less than $\varepsilon/4p$ and the same is done with each of the p triangles we have. The union of ω_1, ω_2 and all the small triangles of the p Perron trees has an area less than $\varepsilon/4 + p\,\varepsilon/4p = \varepsilon/2$. One of these small triangles composing one of the Perron trees has the situation indicated in Figure 15 (it has been enlarged to make the figure more easily understandable).

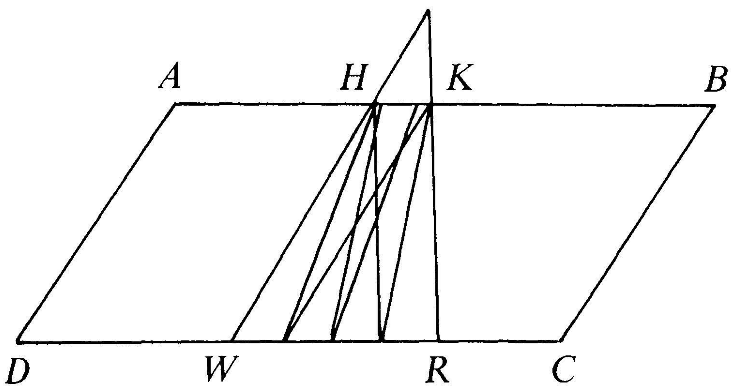

FIGURE 15

We wish to substitute the trapezoid $HKRW$ by strips, what we can easily do without augmenting the area in the way schematically indicated in Figure 15, where four parallelograms have sufficed to cover the portion of $ABCD$ covered by $HKRW$.

So we obtain ω_1, ω_2 and a number of small strips covering together an area smaller than $\varepsilon/2$. It is easy to see that for each segment joining A to a point of DC, there is another one parallel to it of the same length inside one of the strips we have obtained.

If we now perform an analogous construction starting from the other side BC of $ABCD$ we obtain a finite number of strips satisfying all the properties indicated (a), (b), (c).

The Besicovitch set is now very easily obtained as follows. We take a square $MNPQ$ of side length 1 and apply to it the above auxiliary construction (**) with $\varepsilon = 1/2$. We obtain a number of strips $\omega_1^1, \omega_2^1, \ldots, \omega_{r_1}^1$, covering an area smaller than 1/2 and such that for each segment determined by a point of MN and another of PQ there is a segment of the same length and direction inside $\Omega^1 = \omega_2^1 \cup \omega_2^1 \cup \ldots \cup \omega_{r_1}^1$.

Now we consider each of the parallelograms ω_j^1 and apply to it the same construction (**) with $\varepsilon = 1/2^2 r_1$. Collecting all parallelograms corresponding to each $\omega_j^1, j = 1, 2, \ldots, r_1$, we obtain a second family of parallelograms $\omega_1^2, \omega_2^2, \ldots, \omega_{r_2}^2$. Their union $\Omega^2 = \omega_1^2 \cup \omega_2^2 \cup \ldots \cup \omega_{r_2}^2$ has area less than 1/2, is contained in Ω^1 and, again, for each segment joining a point of MN to another of PQ there is another one of the same length and direction inside Ω^2. We proceed with the parallelograms ω_j^2 as we did with the ω_j^1, now with $\varepsilon = 1/2^3 r_2$, and so on. Thus we obtain

$$\Omega^1 \supset \Omega^2 \supset \Omega^3 \supset \ldots$$

of areas

$$S(\Omega^1) < 1/2,\ S(\Omega^2) < 1/2^2,\ S(\Omega^3) < 1/2^3, \ldots$$

The sets Ω^j are compact and have the property of containing a parallel translation of each segment with one extremity on MN and the other on PQ. The intersection

$$B = \Omega^1 \cap \Omega^2 \cap \ldots \cap \Omega^j \cap \ldots$$

is of null area and has this same property. We now proceed with the square $MNPQ$ in the same way in the other direction and obtain a compact set of measure zero containing a segment of length one in each direction, i.e. the Besicovitch set.

8. The Nikodym set

The Nikodym set can be obtained from the Perron tree in a similar way through the following auxiliary construction, also surprising in itself.

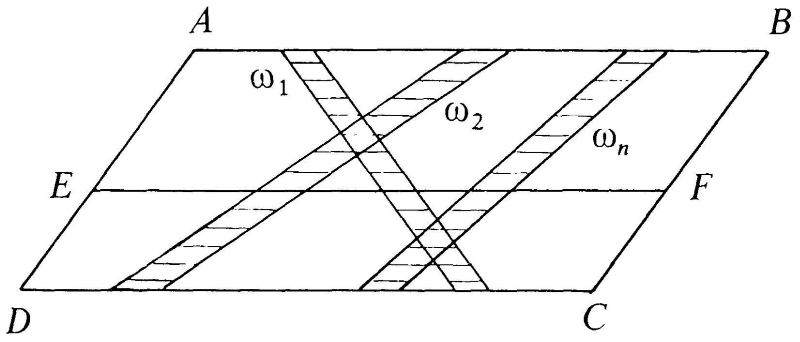

FIGURE 16

Let $ABCD$ be an arbitrary parallelogram and $CDEF$ another one contained in it as Figure 16 shows. Let ε be any arbitrary positive number. Then one can construct a finite number of parallelograms $\omega_1, \omega_2, ..., \omega_q$, with a basis on CD and another one on AB such that the figure $\omega_1 \cup \omega_2 \cup ... \cup \omega_q$ covers $CDEF$ while the part of it above EF has area less than ε, that is:

$$\omega_1 \cup \omega_2 \cup ... \cup \omega_q \supset CDEF$$
$$S((\omega_1 \cup \omega_2 \cup ... \cup \omega_q) \cap (ABCD - CDEF)) < \varepsilon$$

This construction is a little more technical than that of the Besicovitch set and will be omitted. For details we refer to Guzmán (1975).

9. MATHEMATICAL FRIVOLITIES? FROM THE PERRON TREE TO THE MEASURE OF THE DENSITY

What started as a puzzle has proved to have many important applications to solve some interesting problems of recent analysis.

Let us assume that we have a mass distributed on the plane and that we wish to measure the density of this distribution at each point. Let us also suppose that the mass is not continuously distributed. One can perhaps say: "Will it not be very artificial to consider a mass that is not continuously distributed?" It is true that the old Scholastic used to affirm that "*natura non facit saltus*" (nature does not proceed by jumps). However, the findings of modern physics permit us to affirm with even stronger motivation "*natura non facit nisi saltus*" (nature proceeds only by jumps). Therefore it is rather natural to consider a discontinuous mass distribution.

For a long time one thought that in order to measure the density one could take *any system of reasonable sets* that contract to the point at which one measures the density, find the mean density over such sets and hope that, when the sets become smaller and smaller, the mean density approaches a number, the density at the point. The Nikodym set shows in an easy way that one has to be very careful at choosing *reasonable sets.* As Zygmund observed (see the end of Nikodym's paper in 1927), it follows from the Nikodym set that if we take something apparently so reasonable as the system of all rectangles centered at the corresponding points, the mean densities can diverge. This, however, does not happen if the system is that of all circles or squares containing the points. Considerations of this type have given rise to the modern theory of differentiation of integrals.

10. ANOTHER FRUIT OF THE PERRON TREE. A PROBLEM ON DOUBLE FOURIER SERIES

A famous problem in Fourier analysis, open for a long time, has been recently solved in a rather simple way be the use of the Perron tree.

For a periodic function of two variables $f(x, y)$ of period 1 in each variable one can define its Fourier coefficients setting for $m = 0, \pm 1, \pm 2, \ldots, n = 0, \pm 1, \pm 2, \ldots$

$$a_{mn} = \int_0^1 \int_0^1 f(x, y) e^{-2\pi i m x} e^{-2\pi i n y} \, dx \, dy$$

and one can construct the corresponding Fourier series

$$\sum_{m,n} a_{mn} e^{2\pi i m x} e^{2\pi i n y} .$$

One can consider the partial sums of this series in several ways, in order to explore whether they converge or not to the original function. Thus, for example, one can consider the "square" sums

$$S_P f(x, y) = \sum_{\substack{|m| < P \\ |n| < P}} a_{mn} e^{2\pi i m x} e^{2\pi i n y}$$

or else the "rectangular" sums

$$S_{M,N} f(x, y) = \sum_{\substack{|m| < M \\ |n| < N}} a_{mn} e^{2\pi i m x} e^{2\pi i n y}$$

and examine whether in some sense $S_P f \to f$ as $P \to \infty$ or $S_{M,N} f \to f$ as $M, N \to \infty$. One can also consider the "circular" sums

$$S^R f(x, y) = \sum_{m^2 + n^2 \leqslant R^2} a_{mn} e^{2\pi i m x} e^{2\pi i n y}$$

As a consequence of the construction of the Perron tree one can prove, for example, that there are functions $f \in L^p$, $1 < p < 2$, such that $S^R f(x, y)$ diverges at almost each point (x, y). For this result we refer to the papers by C. Fefferman in 1970 and 1971.

Our short excursion comes to confirm what happens so often in Mathematics. Apparently idle and superfluous questions give rise to very interesting and important portions of mathematics, useful in many respects. As Littlewood used to say, a good mathematical game is worth many theorems.

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