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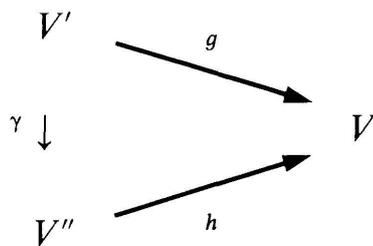
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$\mathcal{S}_{\text{Alg}}(V)$  is the set of distinct algebraic structures on  $V$ . Hence a natural problem is to compute  $\mathcal{S}_{\text{Alg}}(V)$ , or at least produce nontrivial elements of this set. For example if we take  $M \subset V$  as in Proposition 2.10, then by Theorem 2.12  $(V, M)$  is diffeomorphic to nonsingular algebraic sets  $(V', M')$ . Let  $|V| = |V'|$  denote the underlying smooth structures and let  $V \xrightarrow{g} |V|$ ,  $V' \xrightarrow{g'} |V|$  be the forgetful maps. Then  $(V, g)$  and  $(V', g')$  are distinct elements of  $\mathcal{S}_{\text{Alg}}(|V|)$ , otherwise  $M$  would be isotopic to a nonsingular algebraic subset of  $V$ .

An interesting question is whether algebraic structures on smooth manifolds satisfy the product structure theorem; that is, whether the natural map

$$\mathcal{S}_{\text{Alg}}(M) \times \mathbf{R}^n \rightarrow \mathcal{S}_{\text{Alg}}(M \times \mathbf{R}^n), (V, g) \mapsto (V \times \mathbf{R}^n, g \times id)$$

is surjection. The answer would be negative if one can find a smooth manifold  $M$  and  $\theta \in H_*(M; \mathbf{Z}/2\mathbf{Z})$  such that  $M$  can not be diffeomorphic to a nonsingular algebraic set  $M'$  with  $\theta \in H_*^A(M'; \mathbf{Z}/2\mathbf{Z})$ . To see this, pick any smooth representative  $N \xrightarrow{g} M$  of  $\theta = g_*[N]$ . By graphing  $g$ , we can assume  $N \subset M \times \mathbf{R}^n$  for some  $n$  and  $g$  is induced by projection. By Theorem 2.12 we can find a diffeomorphism  $\lambda : M \times \mathbf{R}^n \rightarrow V$  to a nonsingular algebraic set  $V$  with  $\lambda(N)$  is an algebraic subset (one has to modify Theorem 2.12 to apply to this noncompact case). Then there can not exist a birational diffeomorphism  $\mu : V \rightarrow M' \times \mathbf{R}^n$  where  $M'$  is a nonsingular algebraic set diffeomorphic to  $M$ , otherwise  $\lambda(N) \xrightarrow{\mu} M' \times \mathbf{R}^n \xrightarrow{\text{projection}} M'$  would represent  $\theta \in H_*^A(M'; \mathbf{Z}/2\mathbf{Z})$ .

### §3. BLOWING DOWN

Real algebraic sets obey some simple but useful topological properties:

PROPOSITION 3.1.

- (a) *One point compactification an algebraic set is homeomorphic to an algebraic set.*
- (b) *Given algebraic sets  $L \subset V$ , then  $V - L$  is homeomorphic to an algebraic set.*
- (c) *Given algebraic sets  $L \subset V$  with  $V$  compact then  $V/L$  is homeomorphic to an algebraic set.*

*Proof:*

(a) Let  $Z \subset \mathbf{R}^n$  be an algebraic set and assume that  $Z \neq \mathbf{R}^n$  and  $0 \notin Z$  (otherwise translate  $Z$ ). Let  $Z = f^{-1}(0)$  for some polynomial  $f(x)$ ; then define

$F(x) = |x|^{2d} f\left(\frac{x}{|x|^2}\right)$ , where  $d$  is the degree of  $f(x)$ . Clearly  $F(x)$  is a poly-

nomial and  $F^{-1}(0)$  is the one point compactification of  $Z$ , since  $x \mapsto \frac{x}{|x|^2}$  is the inversion through the unit sphere.

(b) Let  $V = f^{-1}(0)$ ,  $L = g^{-1}(0)$  for some polynomials  $f, g: \mathbf{R}^n \rightarrow \mathbf{R}$ . Define  $G(x, t) = |f(x)|^2 + |tg(x) - 1|^2$ , then  $G^{-1}(0) \approx V - L$ .

(c) By applying (a) we get the one point compactification of  $G^{-1}(0)$  to be an algebraic set; if  $V$  is compact this set is homeomorphic to  $V/L$ .  $\square$

This proposition implies that a set is homeomorphic to an algebraic set if and only if the one point compactification is homeomorphic to an algebraic set. Hence any noncompact algebraic set has a collar at infinity, since every algebraic set is locally cone-like [M]. Also we get that the reduced suspension  $\Sigma^n X = X \times S^n / X \vee S^n$  of any algebraic set  $X$  is homeomorphic to an algebraic set.

There is a fancier version of the blowing down operation (c) (Proposition 3.3). First we need to discuss projectively closed algebraic sets. Let  $p: \mathbf{R}^n \rightarrow \mathbf{R}$  be a polynomial. Another interpretation of this concept is the following: Let  $\lambda: \mathbf{R}^n \rightarrow \mathbf{R}^d$ . We call  $p(x)$  an *overt polynomial* if  $p_d^{-1}(0)$  is either the empty set or  $\{0\}$ . We call an algebraic set  $V = p^{-1}(0)$  a *projectively closed algebraic set* if  $p(x)$  is an overt polynomial. Another interpretation of this concept is the following: Let  $\lambda: \mathbf{R}^n \rightarrow \mathbf{RP}^n$  be the inclusion  $\lambda(x_1, \dots, x_n) = [1; x_1; \dots; x_n]$  then  $V = p^{-1}(0)$  is projectively closed if and only if  $\lambda$  is a projective algebraic subset of  $\mathbf{RP}^n$  in other words  $\lambda(V)$  is Zariski closed in  $\mathbf{RP}^n$  (see also [AK<sub>2</sub>]). Real algebraic sets along with maps can easily be made projectively closed by the following.

**PROPOSITION 3.2.** *Let  $f: Z \rightarrow W$  be an entire rational function between algebraic sets with  $Z$  nonsingular and compact. Then there is a projectively closed algebraic set  $V \subset W \times \mathbf{R}^n$  a birational diffeomorphism  $g$  which makes the following commute*

$$\begin{array}{ccc} V & \hookrightarrow & W \times \mathbf{R}^n \\ g \uparrow \approx & & \downarrow \pi \\ Z & \xrightarrow{f} & W \end{array}$$

where  $\pi$  is the projection,  $n$  is some integer.

*Proof:* By taking the graph of  $f$  we can assume that  $Z \subset W \times \mathbf{R}^m \subset \mathbf{R}^r$  for some  $r$ , and  $f$  is induced by projection. Also identify  $\mathbf{R}^r \subset \mathbf{RP}^r$  via  $\lambda$ . Then let  $\bar{Z}$  be the Zariski closure of  $Z$  in  $\mathbf{RP}^r$ . We claim  $\dim(\bar{Z} - Z) < \dim(Z)$ . This is because if  $U$  is an irreducible component of  $\bar{Z}$  then  $U \cap Z \neq \emptyset$ , and therefore  $U - Z = U \cap \mathbf{RP}^{r-1}$  is a proper algebraic subset of  $U$  where  $\mathbf{RP}^{r-1} = \{[0; x_1; \dots; x_r] \in \mathbf{RP}^r\}$ . Since  $U$  is irreducible  $\dim(U - Z) < \dim(U)$ , also  $\dim(U) = \dim(Z)$ . Therefore  $\dim(\bar{Z} - Z) < \dim(Z)$ . So  $\bar{Z} - Z = \text{Sing}(\bar{Z})$ . By resolution of singularities [H] (Theorem 1.1) there is a nonsingular algebraic set  $V \subset \mathbf{RP}^r \times \prod_i \mathbf{RP}^{a_i}$  such that the projection induces birational diffeomorphism between  $V$  and  $Z$ . In particular  $V \subset \mathbf{R}^r \times \prod_i \mathbf{RP}^{a_i}$ .

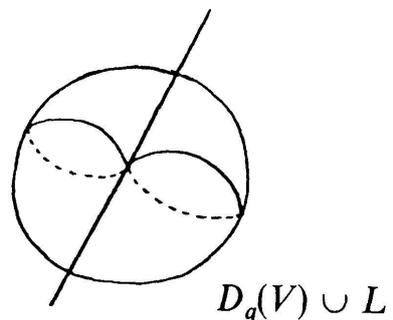
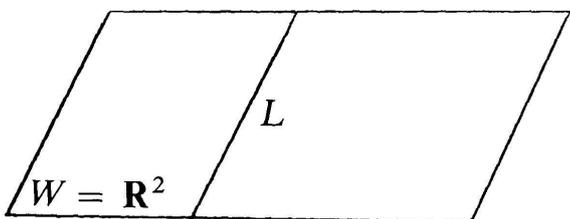
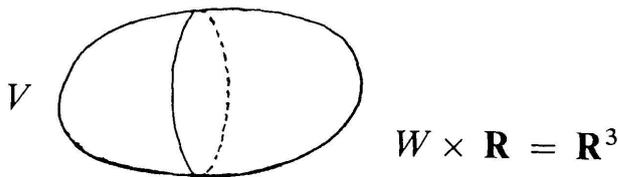
$$\mathbf{RP}^r \times \prod_i \mathbf{RP}^{a_i} \hookrightarrow \mathbf{R}^{(r+1)^2 + \sum(a_i+1)^2}$$

is a projectively closed algebraic set. Hence  $V$  is projectively closed (check details). □

Now assume that  $L \subset W \subset \mathbf{R}^m$  be real algebraic sets, and  $V \subset W \times \mathbf{R}^n$  be a projectively closed algebraic set. Let  $q : \mathbf{R}^m \rightarrow \mathbf{R}$  be a polynomial with  $q^{-1}(0) = L$ . Define

$$D_q : W \times \mathbf{R}^n \rightarrow W \times \mathbf{R}^n$$

by  $D_q(x, y) = (x, yq(x))$ .  $D_q$  is a diffeomorphism on  $(W - L) \times \mathbf{R}^n$  and  $D_q(L \times \mathbf{R}^n) = L \times 0$ . Therefore  $D_q(V)$  is the quotient space of  $V$  by the equivalence relation  $(x, y) \sim (x, 0)$  if  $x \in L$ . We call the operation  $V \rightarrow D_q(V) \cup L$  ( $L$  is identified by  $L \times 0$ ) *blowing down  $V$  over  $L$* .



PROPOSITION 3.3. Given  $L, W, V$  as above, then  $D_q(V) \cup L$  is an algebraic subset of  $W \times \mathbf{R}^n$ .

*Proof:* Let  $p: \mathbf{R}^m \times \mathbf{R}^n \rightarrow \mathbf{R}$  be an overt polynomial of degree  $e$  with  $V = p^{-1}(0)$  and let  $q$  be as above. Define a polynomial  $r: \mathbf{R}^m \times \mathbf{R}^n \rightarrow \mathbf{R}$  by

$$r(x, y) = q(x)^e p\left(x, \frac{y}{q(x)}\right)$$

We claim  $r^{-1}(0) = D_q(V) \cup L$ . It is easy to see that

$$r^{-1}(0) \cap (W - L) \times \mathbf{R}^n = D_q(V) \cap (W - L) \times \mathbf{R}^n,$$

so it suffices to show that  $r^{-1}(0) \cap (L \times \mathbf{R}^n) = L \times 0$ . We decompose  $p(x, y) = p_e(x, y) + \alpha(x, y)$  where  $p_e(x, y)$  is homogeneous of degree  $e$  and  $\alpha(x, y)$  is a polynomial of degree less than  $e$ . Hence if  $(x, y) \in r^{-1}(0) \cap (L \times \mathbf{R}^n)$  then  $r(x, y) = 0$  and  $q(x) = 0$ , which implies  $r(x, y) = p_e(0, y) = 0$ . Then  $y = 0$  since  $p$  is overt, so  $(x, y) \in L \times 0$ . Conversely if  $(x, y) \in L \times 0$  then  $y = 0$  and  $q(x) = 0$ . Hence  $r(x, y) = p_e(0, 0) = 0$ , i.e.  $(x, y) \in r^{-1}(0) \cap (L \times \mathbf{R}^n)$ .  $\square$

There is a more useful version of Proposition 3.3 which says that after modifying  $D_q$  we can get  $D_q(V) \cup L$  to be a projectively closed algebraic set (Proposition 3.1 of [AK<sub>6</sub>]). This allows us to iterate this blowing down process.

#### §4. ISOLATED SINGULARITIES

The topology of real algebraic sets with isolated singularities is completely understood by the following Theorem.

THEOREM 4.1 ([AK<sub>2</sub>]).  $X$  is homeomorphic to an algebraic set with isolated singularities if and only if  $X$  is obtained by taking a smooth compact manifold  $W$  with boundary  $\partial W = \bigcup_{i=1}^r M_i$ , where each  $M_i$  bounds, then crushing some  $M_i$ 's to points and deleting the remaining  $M_i$ 's.

