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$$\begin{array}{ccc} V' & \searrow^{g} & \\ \gamma\downarrow & & V \\ V'' & \nearrow_{h} & \end{array}$$

$\mathcal{S}_{\mathrm{Alg}}(V)$ is the set of distinct algebraic structures on V. Hence a natural problem is to compute $\mathcal{S}_{\mathrm{Alg}}(V)$, or at least produce nontrivial elements of this set. For example if we take $M \subset V$ as in Proposition 2.10, then by Theorem 2.12 (V, M) is diffeomorphic to nonsingular algebraic sets (V', M'). Let $|V| = |V'|$ denote the underlying smooth structures and let $V \xrightarrow{g} |V|$, $V' \xrightarrow{g'} |V|$ be the forgetful maps. Then (V, g) and (V', g') are distinct elements of $\mathcal{S}_{\mathrm{Alg}}(|V|)$, otherwise M would be isotopic to a nonsingular algebraic subset of V.

An interesting question is whether algebraic structures on smooth manifolds satisfy the product structure theorem; that is, whether the natural map

$$\mathcal{S}_{\mathrm{Alg}}(M) \times \mathbf{R}^n \to \mathcal{S}_{\mathrm{Alg}}(M \times \mathbf{R}^n), \ (V, g) \mapsto (V \times \mathbf{R}^n, g \times id)$$

is surjection. The answer would be negative if one can find a smooth manifold M and $\theta \in H_*(M; \mathbf{Z}/2\mathbf{Z})$ such that M can not be diffeomorphic to a nonsingular algebraic set M' with $\theta \in H_*^A(M'; \mathbf{Z}/2\mathbf{Z})$. To see this, pick any smooth representative $N \xrightarrow{g} M$ of $\theta = g_*[N]$. By graphing g, we can assume $N \subset M \times \mathbf{R}^n$ for some n and g is induced by projection. By Theorem 2.12 we can find a diffeomorphism $\lambda : M \times \mathbf{R}^n \to V$ to a nonsingular algebraic set V with $\lambda(N)$ is an algebraic subset (one has to modify Theorem 2.12 to apply to this noncompact case). Then there can not exist a birational diffeomorphism $\mu : V \to M' \times \mathbf{R}^n$ where M' is a nonsingular algebraic set diffeomorphic to M, otherwise $\lambda(N) \xrightarrow{\mu} M' \times \mathbf{R}^n \xrightarrow{\text{projection}} M'$ would represent $\theta \in H_*^A(M'; \mathbf{Z}/2\mathbf{Z})$.

§3. Blowing Down

Real algebraic sets obey some simple but useful topological properties:

PROPOSITION 3.1.

(a) *One point compactification an algebraic set is homeomorphic to an algebraic set.*

(b) *Given algebraic sets $L \subset V$, then $V - L$ is homeomorphic to an algebraic set.*

(c) *Given algebraic sets $L \subset V$ with V compact then V/L is homeomorphic to an algebraic set.*

Proof:

(a) Let $Z \subset \mathbf{R}^n$ be an algebraic set and assume that $Z \neq \mathbf{R}^n$ and $0 \notin Z$ (otherwise translate Z). Let $Z = f^{-1}(0)$ for some polynomial $f(x)$; then define $F(x) = |x|^{2d} f\left(\frac{x}{|x|^2}\right)$, where d is the degree of $f(x)$. Clearly $F(x)$ is a polynomial and $F^{-1}(0)$ is the one point compactification of Z, since $x \mapsto \frac{x}{|x|^2}$ is the inversion through the unit sphere.

(b) Let $V = f^{-1}(0)$, $L = g^{-1}(0)$ for some polynomials $f, g : \mathbf{R}^n \to \mathbf{R}$. Define $G(x, t) = |f(x)|^2 + |tg(x) - 1|^2$, then $G^{-1}(0) \approx V - L$.

(c) By applying (a) we get the one point compactification of $G^{-1}(0)$ to be an algebraic set; if V is compact this set is homeomorphic to V/L. □

This proposition implies that a set is homeomorphic to an algebraic set if and only if the one point compactification is homeomorphic to an algebraic set. Hence any noncompact algebraic set has a collar at infinity, since every algebraic set is locally cone-like [M]. Also we get that the reduced suspension $\Sigma^n X = X \times S^n / X \vee S^n$ of any algebraic set X is homeomorphic to an algebraic set.

There is a fancier version of the blowing down operation (c) (Proposition 3.3). First we need to discuss projectively closed algebraic sets. Let $p : \mathbf{R}^n \to \mathbf{R}$ be a polynomial. Another interpretation of this concept is the following: Let λ: $\mathbf{R}^n$ d. We call $p(x)$ an *overt polynomial* if $p_d^{-1}(0)$ is either the empty set or $\{0\}$. We call an algebraic set $V = p^{-1}(0)$ a *projectively closed algebraic* set if $p(x)$ is an overt polynomial. Another interpretation of this concept is the following: Let $\lambda : \mathbf{R}^n \to \mathbf{RP}^n$ be the inclusion $\lambda(x_1, \ldots, x_n) = [1; x_1; \ldots; x_n]$ then $V = p^{-1}(0)$ is projectively closed if and only if λ is a projective algebraic subset of $\mathbf{RP}^n$ in other words $\lambda(V)$ is Zariski closed in $\mathbf{RP}^n$ (see also [AK_2]). Real algebraic sets along with maps can easily be made projectively closed by the following.

PROPOSITION 3.2. *Let $f : Z \to W$ be an entire rational function between algebraic sets with Z nonsingular and compact. Then there is a projectively closed algebraic set $V \subset W \times \mathbf{R}^n$ a birational diffeomorphism g which makes the following commute*

$$\begin{array}{ccc} V & \hookrightarrow & W \times \mathbf{R}^n \\ {}^{g}\uparrow \approx & & \downarrow {}^{\pi} \\ Z & \xrightarrow{f} & W \end{array}$$

where π is the projection, n is some integer.

Proof: By taking the graph of f we can assume that $Z \subset W \times \mathbf{R}^m \subset \mathbf{R}^r$ for some r, and f is induced by projection. Also identify $\mathbf{R}^r \subset \mathbf{RP}^r$ via λ. Then let $\bar{Z}$ be the Zariski closure of Z in $\mathbf{RP}^r$. We claim $\dim(\bar{Z}-Z) < \dim(Z)$. This is because if U is an irreducible component of $\bar{Z}$ then $U \cap Z \neq \emptyset$, and therefore $U - Z = U \cap \mathbf{RP}^{r-1}$ is a proper algebraic subset of U where $\mathbf{RP}^{r-1} = \{[0; x_1; \ldots; x_r] \in \mathbf{RP}^r\}$. Since U is irreducible $\dim(U-Z) < \dim(U)$, also $\dim(U) = \dim(Z)$. Therefore $\dim(\bar{Z}-Z) < \dim(Z)$. So $\bar{Z} - Z = \mathrm{Sing}(\bar{Z})$. By resolution of singularities [H] (Theorem 1.1) there is a nonsingular algebraic set $V \subset \mathbf{RP}^r \times \prod_i \mathbf{RP}^{a_i}$ such that the projection induces birational diffeomorphism between V and Z. In particular $V \subset \mathbf{R}^r \times \prod_i \mathbf{RP}^{a_i}$.

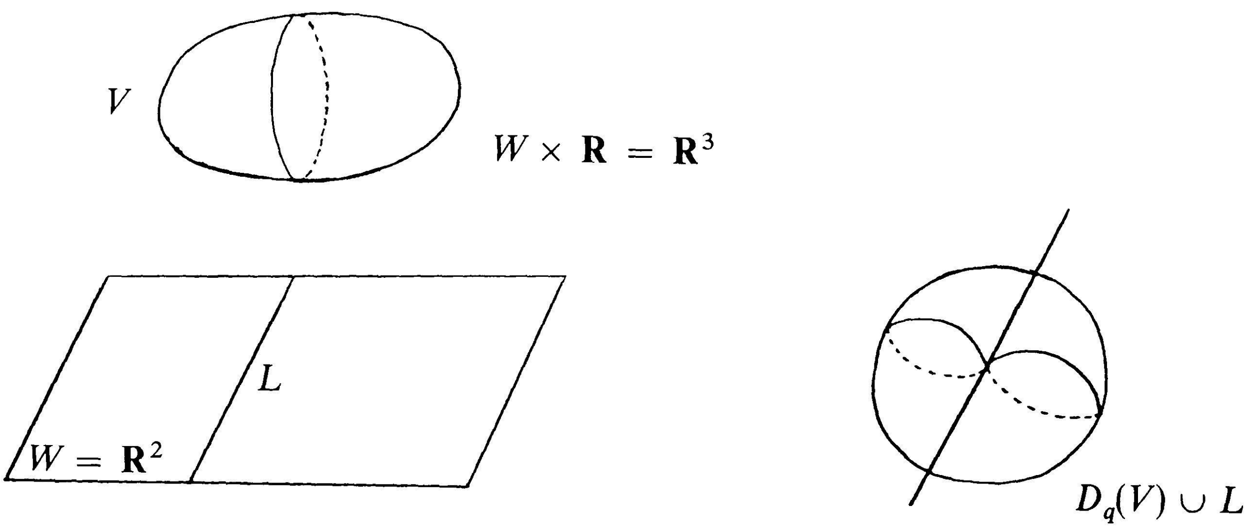
$$\mathbf{RP}^r \times \prod_i \mathbf{RP}^{a_i} \hookrightarrow \mathbf{R}^{(r+1)^2 + \Sigma(a_i+1)^2}$$

is a projectively closed algebraic set. Hence V is projectively closed (check details). □

Now assume that $L \subset W \subset \mathbf{R}^m$ be real algebraic sets, and $V \subset W \times \mathbf{R}^n$ be a projectively closed algebraic set. Let $q : \mathbf{R}^m \to \mathbf{R}$ be a polynomial with $q^{-1}(0) = L$. Define

$$D_q : W \times \mathbf{R}^n \to W \times \mathbf{R}^n$$

by $D_q(x, y) = (x, yq(x))$. D_q is a diffeomorphism on $(W-L) \times \mathbf{R}^n$ and $D_q(L \times \mathbf{R}^n) = L \times 0$. Therefore $D_q(V)$ is the quotient space of V by the equivalence relation $(x, y) \sim (x, 0)$ if $x \in L$. We call the operation $V \to D_q(V) \cup L$ (L is identified by $L \times 0$) *blowing down* V *over* L.



PROPOSITION 3.3. *Given* L, W, V *as above, then* $D_q(V) \cup L$ *is an algebraic subset of* $W \times \mathbf{R}^n$.

Proof: Let $p : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}$ be an overt polynomial of degree e with $V = p^{-1}(0)$ and let q be as above. Define a polynomial $r : \mathbf{R}^m \times \mathbf{R}^n \to \mathbf{R}$ by

$$r(x, y) = q(x)^e p\left(x, \frac{y}{q(x)}\right)$$

We claim $r^{-1}(0) = D_q(V) \cup L$. It is easy to see that

$$r^{-1}(0) \cap (W-L) \times \mathbf{R}^n = D_q(V) \cap (W-L) \times \mathbf{R}^n,$$

so it suffices to show that $r^{-1}(0) \cap (L \times \mathbf{R}^n) = L \times 0$. We decompose $p(x, y) = p_e(x, y) + \alpha(x, y)$ where $p_e(x, y)$ is homogeneous of degree e and $\alpha(x, y)$ is a polynomial of degree less than e. Hence if $(x, y) \in r^{-1}(0) \cap (L \times \mathbf{R}^n)$ then $r(x, y) = 0$ and $q(x) = 0$, which implies $r(x, y) = p_e(0, y) = 0$. Then $y = 0$ since p is overt, so $(x, y) \in L \times 0$. Conversely if $(x, y) \in L \times 0$ then $y = 0$ and $q(x) = 0$. Hence $r(x, y) = p_e(0, 0) = 0$, i.e. $(x, y) \in r^{-1}(0) \cap (L \times \mathbf{R}^n)$. □

There is a more useful version of Proposition 3.3 which says that after modifying D_q we can get $D_q(V) \cup L$ to be a projectively closed algebraic set (Proposition 3.1 of [AK_6]). This allows us to iterate this blowing down process.

§4. ISOLATED SINGULARITIES

The topology of real algebraic sets with isolated singularities is completely understood by the following Theorem.

THEOREM 4.1 ([AK_2]). *X is homeomorphic to an algebraic set with isolated singularities if and only if X is obtained by taking a smooth compact manifold W with boundary $\partial W = \bigcup_{i=1}^{r} M_i$, where each M_i bounds, then crushing some M_i's to points and deleting the remaining M_i's.*

$X =$