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Autor: Akbulut, Selman / King, Henry
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$$\begin{array}{ccc}
 V' & \xrightarrow{g} & V \\
 \gamma \downarrow & & \\
 V'' & \xrightarrow{h} & V
 \end{array}$$

$\mathcal{S}_{\text{Alg}}(V)$ is the set of distinct algebraic structures on V . Hence a natural problem is to compute $\mathcal{S}_{\text{Alg}}(V)$, or at least produce nontrivial elements of this set. For example if we take $M \subset V$ as in Proposition 2.10, then by Theorem 2.12 (V, M) is diffeomorphic to nonsingular algebraic sets (V', M') . Let $|V| = |V'|$ denote the underlying smooth structures and let $V \xrightarrow{g} |V|$, $V' \xrightarrow{g'} |V|$ be the forgetful maps. Then (V, g) and (V', g') are distinct elements of $\mathcal{S}_{\text{Alg}}(|V|)$, otherwise M would be isotopic to a nonsingular algebraic subset of V .

An interesting question is whether algebraic structures on smooth manifolds satisfy the product structure theorem; that is, whether the natural map

$$\mathcal{S}_{\text{Alg}}(M) \times \mathbf{R}^n \rightarrow \mathcal{S}_{\text{Alg}}(M \times \mathbf{R}^n), (V, g) \mapsto (V \times \mathbf{R}^n, g \times \text{id})$$

is surjection. The answer would be negative if one can find a smooth manifold M and $\theta \in H_*(M; \mathbf{Z}/2\mathbf{Z})$ such that M can not be diffeomorphic to a nonsingular algebraic set M' with $\theta \in H_*^A(M'; \mathbf{Z}/2\mathbf{Z})$. To see this, pick any smooth representative $N \xrightarrow{g} M$ of $\theta = g_*[N]$. By graphing g , we can assume $N \subset M \times \mathbf{R}^n$ for some n and g is induced by projection. By Theorem 2.12 we can find a diffeomorphism $\lambda: M \times \mathbf{R}^n \rightarrow V$ to a nonsingular algebraic set V with $\lambda(N)$ is an algebraic subset (one has to modify Theorem 2.12 to apply to this noncompact case). Then there can not exist a birational diffeomorphism $\mu: V \rightarrow M' \times \mathbf{R}^n$ where M' is a nonsingular algebraic set diffeomorphic to M , otherwise $\lambda(N) \xrightarrow{\mu} M' \times \mathbf{R}^n \xrightarrow{\text{projection}} M'$ would represent $\theta \in H_*^A(M'; \mathbf{Z}/2\mathbf{Z})$.

§3. BLOWING DOWN

Real algebraic sets obey some simple but useful topological properties:

PROPOSITION 3.1.

- (a) One point compactification an algebraic set is homeomorphic to an algebraic set.
- (b) Given algebraic sets $L \subset V$, then $V - L$ is homeomorphic to an algebraic set.
- (c) Given algebraic sets $L \subset V$ with V compact then V/L is homeomorphic to an algebraic set.

Proof:

(a) Let $Z \subset \mathbf{R}^n$ be an algebraic set and assume that $Z \neq \mathbf{R}^n$ and $0 \notin Z$ (otherwise translate Z). Let $Z = f^{-1}(0)$ for some polynomial $f(x)$; then define $F(x) = |x|^{2d} f\left(\frac{x}{|x|^2}\right)$, where d is the degree of $f(x)$. Clearly $F(x)$ is a polynomial and $F^{-1}(0)$ is the one point compactification of Z , since $x \mapsto \frac{x}{|x|^2}$ is the inversion through the unit sphere.

(b) Let $V = f^{-1}(0)$, $L = g^{-1}(0)$ for some polynomials $f, g: \mathbf{R}^n \rightarrow \mathbf{R}$. Define $G(x, t) = |f(x)|^2 + |tg(x) - 1|^2$, then $G^{-1}(0) \approx V/L$.

(c) By applying (a) we get the one point compactification of $G^{-1}(0)$ to be an algebraic set; if V is compact this set is homeomorphic to V/L . \square

This proposition implies that a set is homeomorphic to an algebraic set if and only if the one point compactification is homeomorphic to an algebraic set. Hence any noncompact algebraic set has a collar at infinity, since every algebraic set is locally cone-like [M]. Also we get that the reduced suspension $\Sigma^n X = X \times S^n / X \vee S^n$ of any algebraic set X is homeomorphic to an algebraic set.

There is a fancier version of the blowing down operation (c) (Proposition 3.3). First we need to discuss projectively closed algebraic sets. Let $p: \mathbf{R}^n \rightarrow \mathbf{R}$ be a polynomial. Another interpretation of this concept is the following: Let $\lambda: \mathbf{R}^n \rightarrow \mathbf{R}^d$. We call $p(x)$ an *overt polynomial* if $p_d^{-1}(0)$ is either the empty set or $\{0\}$. We call an algebraic set $V = p^{-1}(0)$ a *projectively closed algebraic set* if $p(x)$ is an overt polynomial. Another interpretation of this concept is the following: Let $\lambda: \mathbf{R}^n \rightarrow \mathbf{RP}^n$ be the inclusion $\lambda(x_1, \dots, x_n) = [1; x_1; \dots; x_n]$ then $V = p^{-1}(0)$ is projectively closed if and only if λ is a projective algebraic subset of \mathbf{RP}^n in other words $\lambda(V)$ is Zariski closed in \mathbf{RP}^n (see also [AK₂]). Real algebraic sets along with maps can easily be made projectively closed by the following.

PROPOSITION 3.2. *Let $f: Z \rightarrow W$ be an entire rational function between algebraic sets with Z nonsingular and compact. Then there is a projectively closed algebraic set $V \subset W \times \mathbf{R}^n$ a birational diffeomorphism g which makes the following commute*

$$\begin{array}{ccc} V & \hookrightarrow & W \times \mathbf{R}^n \\ g \uparrow \approx & & \downarrow \pi \\ Z & \xrightarrow{f} & W \end{array}$$

where π is the projection, n is some integer.

Proof: By taking the graph of f we can assume that $Z \subset W \times \mathbf{R}^m \subset \mathbf{R}^r$ for some r , and f is induced by projection. Also identify $\mathbf{R}^r \subset \mathbf{RP}^r$ via λ . Then let \bar{Z} be the Zariski closure of Z in \mathbf{RP}^r . We claim $\dim(\bar{Z} - Z) < \dim(Z)$. This is because if U is an irreducible component of \bar{Z} then $U \cap Z \neq \emptyset$, and therefore $U - Z = U \cap \mathbf{RP}^{r-1}$ is a proper algebraic subset of U where $\mathbf{RP}^{r-1} = \{[0; x_1; \dots; x_r] \in \mathbf{RP}^r\}$. Since U is irreducible $\dim(U - Z) < \dim(U)$, also $\dim(U) = \dim(Z)$. Therefore $\dim(\bar{Z} - Z) < \dim(Z)$. So $\bar{Z} - Z = \text{Sing}(\bar{Z})$. By resolution of singularities [H] (Theorem 1.1) there is a nonsingular algebraic set $V \subset \mathbf{RP}^r \times \prod_i \mathbf{RP}^{a_i}$ such that the projection induces birational diffeomorphism between V and Z . In particular $V \subset \mathbf{R}^r \times \prod_i \mathbf{RP}^{a_i}$.

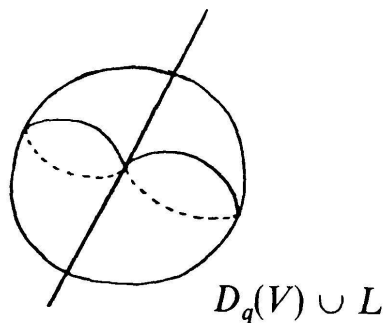
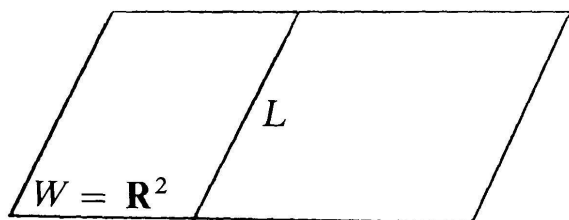
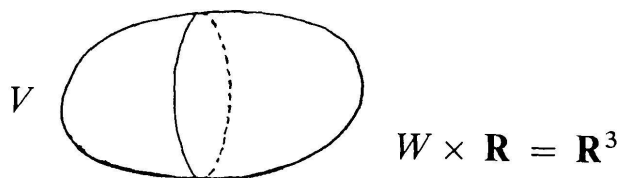
$$\mathbf{RP}^r \times \prod_i \mathbf{RP}^{a_i} \hookrightarrow \mathbf{R}^{(r+1)^2 + \sum (a_i+1)^2}$$

is a projectively closed algebraic set. Hence V is projectively closed (check details). \square

Now assume that $L \subset W \subset \mathbf{R}^m$ be real algebraic sets, and $V \subset W \times \mathbf{R}^n$ be a projectively closed algebraic set. Let $q: \mathbf{R}^m \rightarrow \mathbf{R}$ be a polynomial with $q^{-1}(0) = L$. Define

$$D_q: W \times \mathbf{R}^n \rightarrow W \times \mathbf{R}^n$$

by $D_q(x, y) = (x, yq(x))$. D_q is a diffeomorphism on $(W - L) \times \mathbf{R}^n$ and $D_q(L \times \mathbf{R}^n) = L \times 0$. Therefore $D_q(V)$ is the quotient space of V by the equivalence relation $(x, y) \sim (x, 0)$ if $x \in L$. We call the operation $V \rightarrow D_q(V) \cup L$ (L is identified by $L \times 0$) *blowing down* V over L .



PROPOSITION 3.3. Given L, W, V as above, then $D_q(V) \cup L$ is an algebraic subset of $W \times \mathbf{R}^n$.

Proof: Let $p: \mathbf{R}^m \times \mathbf{R}^n \rightarrow \mathbf{R}$ be an overt polynomial of degree e with $V = p^{-1}(0)$ and let q be as above. Define a polynomial $r: \mathbf{R}^m \times \mathbf{R}^n \rightarrow \mathbf{R}$ by

$$r(x, y) = q(x)^e p\left(x, \frac{y}{q(x)}\right)$$

We claim $r^{-1}(0) = D_q(V) \cup L$. It is easy to see that

$$r^{-1}(0) \cap (W - L) \times \mathbf{R}^n = D_q(V) \cap (W - L) \times \mathbf{R}^n,$$

so it suffices to show that $r^{-1}(0) \cap (L \times \mathbf{R}^n) = L \times 0$. We decompose $p(x, y) = p_e(x, y) + \alpha(x, y)$ where $p_e(x, y)$ is homogeneous of degree e and $\alpha(x, y)$ is a polynomial of degree less than e . Hence if $(x, y) \in r^{-1}(0) \cap (L \times \mathbf{R}^n)$ then $r(x, y) = 0$ and $q(x) = 0$, which implies $r(x, y) = p_e(0, y) = 0$. Then $y = 0$ since p is overt, so $(x, y) \in L \times 0$. Conversely if $(x, y) \in L \times 0$ then $y = 0$ and $q(x) = 0$. Hence $r(x, y) = p_e(0, 0) = 0$, i.e. $(x, y) \in r^{-1}(0) \cap (L \times \mathbf{R}^n)$. \square

There is a more useful version of Proposition 3.3 which says that after modifying D_q we can get $D_q(V) \cup L$ to be a projectively closed algebraic set (Proposition 3.1 of [AK₆]). This allows us to iterate this blowing down process.

§4. ISOLATED SINGULARITIES

The topology of real algebraic sets with isolated singularities is completely understood by the following Theorem.

THEOREM 4.1 ([AK₂]). X is homeomorphic to an algebraic set with isolated singularities if and only if X is obtained by taking a smooth compact manifold W with boundary $\partial W = \bigcup_{i=1}^r M_i$, where each M_i bounds, then crushing some M_i 's to points and deleting the remaining M_i 's.

