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§2. NONSINGULAR ALGEBRAIC SETS

The fact that closed smooth manifolds are diffeomorphic to nonsingular algebraic sets can be traced back to the following simple fact.

PROPOSITION 2.1. *Let* L *be a nonsingular algebraic set and* K *be a compact set with* $L \subset K \subset \mathbf{R}^n$, *let* $f : \mathbf{R}^n \to \mathbf{R}$ *be a smooth function with* $f|_L = u$ *for some entire rational function* u. *Then there is an entire rational function* $p: \mathbf{R}^n \to \mathbf{R}$ *which approximates* f *arbitrarily closely near* K *with* $p|_L = u$ *(if* u *is a polynomial then* p *can be taken to be a polynomial). Furthermore if* $f - u$ *has compact support then* p *can approximate* f *on all of* $\mathbf{R}^n$.

Proof: First write $f - u = \sum_i a_i \,.\, \beta_i$ where a_i are smooth functions and $\beta_i \in I(L)$. Clearly we can do this locally, and then by putting these local expressions together by partitions of unity we get the global expression. We approximate $a_i(x)$ by polynomials $\alpha_i(x)$ near K and let $p = u + \sum_i \alpha_i \,.\, \beta_i$. $p(x)$ has the required properties. If $p - u$ has compact support we can define a smooth function $g: S^n \to \mathbf{R}$ by $g = (f-u) \circ \theta$ on $S^n - (0, 1)$ and $g(0, 1) = 0$, where $S^n \subset \mathbf{R}^n \times \mathbf{R}$ is the unit sphere and $\theta: S^n - (0, 1) \to \mathbf{R}^n$ is the stereographic projection, $\theta(x, t) = \dfrac{x}{1 - t}$. Then

$$g : \left(S^n, \theta^{-1}(L) \cup (0, 1)\right) \to (\mathbf{R}, 0)$$

hence by the first part of the theorem g can be approximated by an entire rational function

$$\hat{p}: \left(S^n, \theta^{-1}(L) \cup (0, 1)\right) \to (\mathbf{R}, 0)\,.$$

Let $p = \hat{p} \circ \theta^{-1} + u$. □

The following was introduced in [AK_2] to simplify Nash's and Tognoli's theorems.

PROPOSITION 2.2 (Normalization). *Given* $L \subset K \subset \mathbf{R}^n$, $W \subset \mathbf{R}^m$ *where* L, W *are nonsingular algebraic sets and* K *is a compact set, and* $f : K \to W$ *a smooth function with* $f|_L = u$ *for some entire rational function* $u : L \to W$. *Then there is an algebraic set* $Z \subset \mathbf{R}^n \times \mathbf{R}^m$ *and an entire rational function*

$p : Z \to W$ and an open neighborhood U of K in $\mathbf{R}^n$ and a smooth function $\varphi : (U, L) \to (\mathbf{R}^m, 0)$ such that

(i) *The set $\tilde{U} = \{(x, \varphi(x)) \mid x \in U\} \subset \mathbf{R}^n \times \mathbf{R}^m$ is an open nonsingular subset of Z.*

(ii) *p is arbitrarily close to $f \circ \pi$ on $\tilde{U}$ where π is the projection to the first factor.*

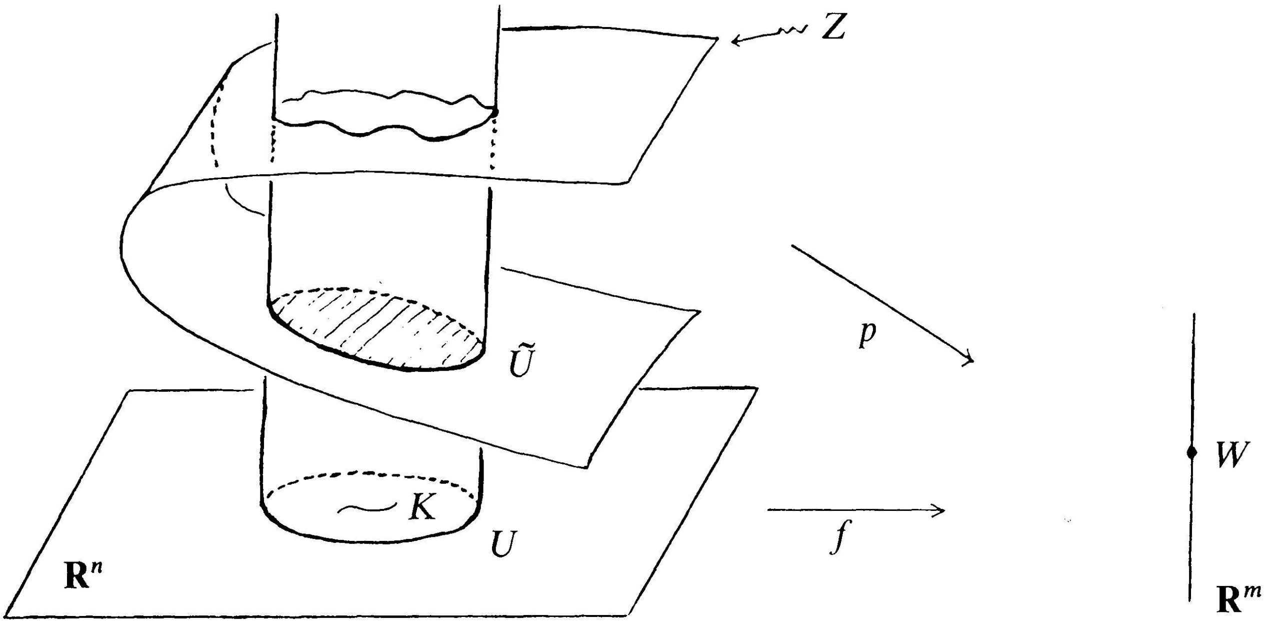
(iii) *$L \times 0 \subset \tilde{U}$ and $p|_{L \times 0} = u$.*

Proof: Let $\delta : \mathbf{R}^m \to \mathbf{R}^{m^2}$ be an entire rational function with

$$\delta(x) \in G(m, m - \dim W)$$

is the normal plane to W at $x \in W$ (from §0). By Proposition 2.1 there is an entire rational function $g : \mathbf{R}^n \to \mathbf{R}^m$ which approximates f on K with $g|_L = u$. Define:

$$Z = \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^m \mid g(x) + y \in W, \delta(g(x) + y)y = y\}$$
$$p : Z \to \mathbf{R}^m, p(x, y) = g(x) + y$$



Clearly Z is an algebraic set. Let U be a small open tubular neighborhood of K such that g is arbitrarily close to f on U. Therefore when $x \in U$ there is a unique closest point $v(x)$ on W to $g(x)$. Define $\varphi(x) = v(x) - g(x)$ to be the vector from $g(x)$ to $v(x)$. Hence $\varphi(x)$ is perpendicular to W at $v(x) = g(x) + \varphi(x)$, so $\varphi(x)$ is the unique "small" solution of the equations

$$\begin{Bmatrix} g(x) + y \in W \\ \delta(g(x) + y)y = y \end{Bmatrix} \text{ which is } \begin{Bmatrix} g(x) + y \in W \\ y \text{ is } \perp \text{ to } W \text{ at } g(x) + y \end{Bmatrix}$$

Hence $\tilde{U} = \{(x, \varphi(x)) \mid x \in U\}$ has the property

$$\tilde{U} = Z \cap U \times \{y \in \mathbf{R}^m \mid |y| < \varepsilon\}$$

for some small $\varepsilon > 0$. Clearly Z, U, p has the required properties. □

THEOREM 2.3 (Generalized Seifert Theorem). *Let* $M^m \subset V^v$ *be a closed smooth submanifold of a nonsingular algebraic set* V, *imbedded with a trivial normal bundle, and let* $L \subset M$ *be a nonsingular algebraic set. Then by an arbitrarily small isotopy* M *is isotopic to a component of a nonsingular algebraic subset of* V *fixing* L.

Proof: Let $V \subset \mathbf{R}^n$ and let W, U be small open neighborhoods of M^m in V^v, and in $\mathbf{R}^n$ respectively. Let $f : W \to \mathbf{R}^{v-m}$ be the trivialization map of the normal bundle of M in V, f is transverse to $0 \in \mathbf{R}^{v-m}$ and $f^{-1}(0) = M$. Then extend f to $f : U \to \mathbf{R}^{v-m}$. Since $f|_L = 0$ by Proposition 2.1 we can approximate f on Closure(U) by a polynomial $F : (\mathbf{R}^n, L) \to (\mathbf{R}^{v-m}, 0)$. By transversality $F^{-1}(0) \cap W$ is isotopic to $f^{-1}(0) \cap W = M$. In general $F^{-1}(0)$ might have extra components outside of U. □

It is interesting to note that in general the extra components of $F^{-1}(0)$ can not be removed, there are homotopy theoretical obstructions [AK_8] (even when $L = \emptyset$).

Remark 2.4. In Theorem 2.3 it is not necessary to assume that L is nonsingular, it suffices to assume that some open neighborhood W of L in M coincides with an open subset of a nonsingular algebraic set. The proof is the same except it requires a slight modification in Proposition 2.1 (see [AK_2]).

THEOREM 2.5 (Generalized Nash theorem). *Let* $M^m \subset \mathbf{R}^n$ *be a closed smooth submanifold, and* $L \subset M$ *be a nonsingular algebraic set. Assume that some open neighborhood* W *of* L *in* M *is an open subset of some nonsingular algebraic set. Then by an arbitrarily small isotopy* M *can be isotoped to a nonsingular component of an algebraic subset of* $\mathbf{R}^n \times \mathbf{R}^s$ *keeping* L *fixed (for some* s*).*

Proof: Let U be an open tubular neighborhood of M in $\mathbf{R}^n$ and $f : U \to E(n, k)$ be the map which classifies the normal bundle of M in U. $f \pitchfork G(n, k)$ and $f^{-1}(G(n, k)) = M$. By using W we can assume $f|_L = u$ for some entire rational function u (see §0). By Proposition 2.2 there is a nonsingular open subset $\tilde{U}$ of an algebraic set $Z \subset \mathbf{R}^n \times \mathbf{R}^s$ for some s, and an entire rational function $p : \tilde{U} \to E(n, k)$ which makes the following commute

$$\begin{tikzcd}
\mathbf{R}^n \times \mathbf{R}^s \arrow[d, "\pi"] & \supset & \tilde{U} \arrow[d, "\pi"] \arrow[dr, "p"] & & & \\
\mathbf{R}^n & \supset & U \arrow[r, "f"] & E(n, k) & \supset & G(n, k)
\end{tikzcd}$$

where π is projection, and $f \circ \pi$ is close to p, and $L \times 0 \subset \tilde{U}$ with $p|_{L \times 0} = u$.

$$\tilde{U} = \{(x, \varphi(x)) \mid x \in U\}$$

for some smooth function $\varphi(x)$. Let $\hat{p}(x) = p(x, \varphi(x))$ then $\hat{p}$ is close to f on U. By transversality $\hat{p}^{-1}(G(n, k)) \cap U$ is isotopic to $f^{-1}(G(n, k)) \cap U = M$ in U. Since π is an isomorphism on $\tilde{U}$ and $p = \hat{p} \circ \pi$,

$$p^{-1}(G(n, k)) \cap \tilde{U} = \pi^{-1}(\hat{p}^{-1}(G(n, k)) \cap U) \approx M .$$

$p^{-1}(G(n, k)) \cap \tilde{U}$ is a component of an algebraic set by construction and nonsingular by transversality, furthermore it contains $L \times 0$. □

Let V be a nonsingular real algebraic set of dimension n. Recall $AH_{n-1}(V; \mathbf{Z}/2\mathbf{Z})$ is the subgroup of $H_{n-1}(V; \mathbf{Z}/2\mathbf{Z})$ generated by nonsingular algebraic subsets. We define

$$H^t_{n-1}(V) = H_{n-1}(V; \mathbf{Z}/2\mathbf{Z})/AH_{n-1}(V; \mathbf{Z}/2\mathbf{Z}) ,$$

which we call the group of *codimension one transcendental cycles.* For any codimension and closed smooth submanifold $M \subset V$ let $\alpha(M)$ be the image of the fundamental homology class $[M]$ under the quotient map.

THEOREM 2.6 ([AK$_8$]). *Any codimension one closed smooth submanifold $M \subset V$ of a nonsingular algebraic set V is isotopic to a nonsingular algebraic subset by an arbitrarily small isotopy if and only if $\alpha(M) = 0$.*

Sketch of proof: For simplicity assume that M has a trivial normal bundle and $[M]$ is represented by a single nonsingular algebraic subset W of V. If $M \cap W = \emptyset$ then $M \cup W$ separates V into two components V_+, V_- with one of them, say V_+, is compact (since M is homologous to W). Let $f : (V, M \cup W) \to (\mathbf{R}, 0)$ be a smooth function with $f > 0$ on V_+ and $f < 0$ on V_-. We can assume that f is transversal to 0 and is constant outside of a compact set containing V_+. By Proposition 2.1 we can approximate f by a polynomial $F : (V, W) \to (\mathbf{R}, 0)$, then by transversality $F^{-1}(0) = M' \cup W$ where M' is isotopic to M. $M' \cup W$ is a nonsingular algebraic set hence M' is a nonsingular algebraic set.

If $M \cap W \neq \emptyset$ then we can find a smooth representative N of $[M]$ with $N \cap M = \emptyset$ and $N \cap W = \emptyset$. By the first part we can isotope N to a nonsingular algebraic set N' by a small isotopy. Hence $N' \cap M = \emptyset$; and since N' is homologous to M by the previous case M is isotopic to a nonsingular algebraic set by a small isotopy.

The proof of the case M does not have a trivial normal bundle is more difficult, we refer the reader to [AK$_8$]. □

Proposition 2.10 implies that $H^{t}_{n-1}(V)$ is nontrivial in general. One of the corollaries of Theorem 2.6 is that codimension one nonsingular algebraic sets can be moved around by isotopies. A natural generalization of this fact is:

THEOREM 2.7 (Algebraic transversality [AK$_8$]). *Let V be a nonsingular algebraic set and $M \subset V$ be a stable algebraic subset. Let N be a smooth subcomplex of V. Then there exists an arbitrarily small isotopy $f_t : M \to V$ with $f_0(M) = M$ and $f_1(M)$ is a stable algebraic subset transverse to N.*

Let $\eta_*(V)$ be the unoriented bordism group of a nonsingular algebraic set V. Let $\eta^A_*(V)$ be the subgroup of $\eta_*(V)$ generated by entire rational maps $f : M \to V$ where M is a compact nonsingular algebraic set. By taking graph of f one easily sees that every element of $\eta^A_*(V)$ has a representative (M, f), where $M \subset V \times \mathbf{R}^n$ is a nonsingular algebraic set for some n, and f is induced by projection.

THEOREM 2.8. *Let $f : M \to V$ be a map from a closed smooth manifold to a nonsingular algebraic set V. Then $(M, f) \in \eta^A_*(V)$ if and only if $f \times 0$ can be approximated by an imbedding onto a nonsingular algebraic subset of $V \times \mathbf{R}^n$ for some n.*

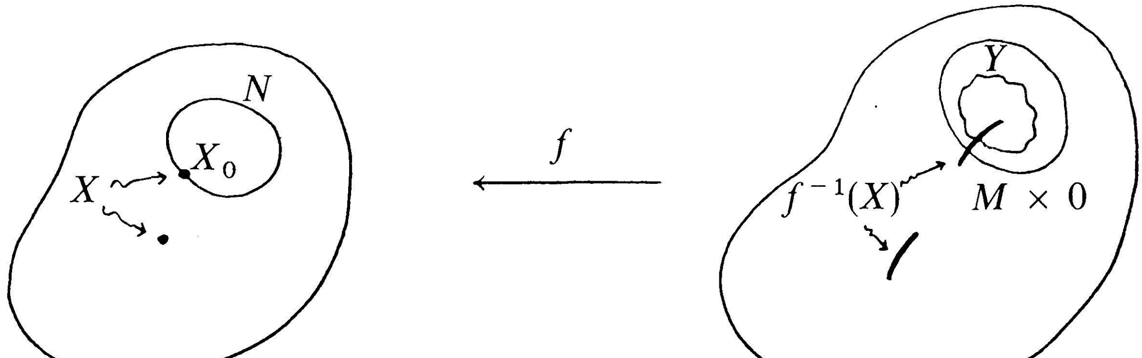
Proof: One way the proof is trivial. Assume $(M, f) \in \eta^A_*(V)$, then there is a smooth manifold Z and a map $F : Z \to V$ with $\partial Z = M \cup N$ and N is a nonsingular algebraic set, $F|_M = f$ and $F|_N$ is an entire rational function. Let $\hat{Z}$ be the double of Z i.e. $\hat{Z} = \partial(Z \times [-1, 1])$. By taking graph of F we may assume $Z \subset V \times \mathbf{R}^s$ is imbedded for some s. In particular $N \subset Z$ is a nonsingular algebraic subset of $V \times \mathbf{R}^s$. Then extend this imbedding to an imbedding $\hat{Z} \subset V \times \mathbf{R}^s \times \mathbf{R}$ which is identity on $N \times (-1, 1)$. Then by Theorem 2.5 we can isotope $\hat{Z}$ to a nonsingular component of an algebraic set $Y \subset V \times \mathbf{R}^n$ for some n with $N \subset Y$. Since the codimension one submanifolds N and M of $\hat{Z}$ are homologous, M can be isotoped to a nonsingular algebraic subset of Y, by Theorem 2.6. □

COROLLARY 2.9 (Tognoli [To]). *Every closed smooth manifold is diffeomorphic to a nonsingular algebraic set.*

The hypothesis of Theorem 2.8 is not void in fact we have:

PROPOSITION 2.10 ([AK$_8$]). *For any k there exist a nonsingular connected algebraic set V and a closed smooth codimension k submanifold $M \subset V$ which can not be isotopic to a nonsingular algebraic subset in $V \times \mathbf{R}^n$ for any n.*

Proof: Let $W = \mathbf{R}^m$ with $m - k$ even, and X be an algebraic subset given by $x_2^4 + (x_1^2-1) . (x_1^2-4) = 0$ and $x_3 = x_4 = \quad = x_m = 0$. X is a nonsingular irreducible algebraic set of two components $X_0 \cup X_1$ each of which is homeomorphic to a circle. Let N be any smooth submanifold of W with $N \cap X = X_0$, and $\dim(N) = m - k$. Then let $M = B(N, X_0)$, $V = B(W, X) \xrightarrow{\pi} W$ be topological and algebraic blowups, respectively. Assume that $M \times 0$ was isotopic to an algebraic subset Y of $V \times \mathbf{R}^n$ by a small isotopy. Then we get a compact nonsingular algebraic set $Z = Y \cap (\pi \circ p)^{-1}(X)$ and an entire rational function $f = \pi \circ p$ where $p\colon V \times \mathbf{R}^n \to V$ is the projection. Furthermore $f\colon Z \to \mathbf{R}^m$ has the properties: $f(Z) = X_0$ and $f^{-1}(x) \approx \mathbf{RP}^{m-k-2}$ for $x \in X_0$ by transversality. Hence since $\bar{X}_0 = X$ and $\chi(\mathbf{RP}^{m-k-2})$ is odd we get a contradiction to Lemma 0.2. ☐



Recall $\eta_*(V) \approx H_*(V; \mathbf{Z}/2\mathbf{Z}) \otimes \eta_*(\text{point})$ and $\eta_*(V)$ is generated by $Q \times N \xrightarrow{\pi} Q \xrightarrow{g} V$ where π is the projection and N is a generator of $\eta_*(\text{point})$ and $g_*[Q]$ is a generator of $H_*(V; \mathbf{Z}/2\mathbf{Z})$. Given $(M, f) \in \eta_*(V)$ with $(M, f) = \Sigma\, \theta \otimes U_i$ then it follows that $(M, f) \in \eta_*^A(V)$ if each $\theta_i \in H_*^A(V; \mathbf{Z}/2\mathbf{Z})$ ([AK$_2$]). If an algebraic set V has the property $H_*(V; \mathbf{Z}/2\mathbf{Z}) = H_*^A(V, \mathbf{Z}/2\mathbf{Z})$ for all * we say that V has *totally algebraic homology;* therefore such algebraic sets have the property $\eta_*(V) = \eta^A_*(V)$. $\mathbf{RP}^m$ and more generally $G(n, m)$ are examples of algebraic sets with totally algebraic homology, because their homology is generated by Schubert cycles. This property is invariant under cross products. Also if $L \subset V$ are nonsingular algebraic sets with totally algebraic homology, then so is $B(V, L)$ (Proposition 6.1 of [AK$_6$]). It is still an open question that whether any closed smooth manifold is diffeomorphic to a nonsingular algebraic set with totally algebraic homology.

Therefore it would be useful to understand when a given homology class $\theta \in H_*(V; \mathbf{Z}/2\mathbf{Z})$ of a nonsingular algebraic set V lies in $H^A_*(V; \mathbf{Z}/2\mathbf{Z})$. This can be detected by a single obstruction $\sigma(\theta)$ as follows. Let $M \subset V$ be a fine submanifold of a nonsingular algebraic set, in particular

$$M = V_0 \subset V_1 \subset \dots \subset V_r \subset V_{r+1} = V$$

for some closed smooth manifolds $\{V_i\}$ with $\dim(V_{i+1}) = \dim(V_i) + 1$, then let

$$\tilde{\alpha}(M) = \operatorname{Inf} \{k \mid \alpha(V_i) = 0 \quad \text{for} \quad i \geqq k\}$$

(make the convention $\alpha(V_{r+1}) = 0$). Recall the definition of $\alpha(V_r) \in H^t_{n-1}(V)$, where $n = \dim(V)$. Theorem 2.6 says that if $\alpha(V_r) = 0$ then V_r can be made a nonsingular algebraic subset of V and therefore $\alpha(V_{r-1}) \in H^t_{n-2}(V_r)$ is defined... etc. Hence by continuing this fashion we see that if $\tilde{\alpha}(M) = 0$ then M is isotopic to an algebraic subset of V.

If $M \subset V$ is just a smooth submanifold of V, then let $\mathscr{F}(V, M)$ be the set of all fine topological multiblowups $\tilde{V} \xrightarrow{\pi} V$ along M $(\mathscr{F}(V, M)) \neq \emptyset$ by Theorem 1.2 and the remarks proceeding it):

$$\tilde{V} = V_k \xrightarrow{\pi_k} V_{k-1} \xrightarrow{\pi_{k-1}} \dots \xrightarrow{\pi_1} V_0 = V,$$

where $V_i = B(V_{i-1}, L_{i-1})$, and $L_i \subset V_i$, $\tilde{M} \subset V_k$ are all fine submanifolds. Make the convention $\tilde{M} = L_k$ then for $(\tilde{V}, \pi) \in \mathscr{F}(V, M)$ define

$$\sigma(\tilde{V}, \pi) = \operatorname{Inf} \{k - n \mid \tilde{\alpha}(L_i) = 0 \quad \text{for} \quad i \leqslant n\}$$

Then $\sigma(\tilde{V}, \pi) = 0$ implies that all $\tilde{\alpha}(L_i) = 0$, hence inductively we can assume that $L_i \subset V_i$ are nonsingular algebraic subsets and therefore we can make $\tilde{V} \xrightarrow{\pi} V$ an algebraic multiblowup and $\tilde{M} \subset \tilde{V}$ an algebraic subset. In fact $\sigma(\tilde{V}, \pi) = 0$ if and only if $\tilde{V} \xrightarrow{\pi} V$ is a stable algebraic multiblowup along M. Let

$$\sigma(M) = \operatorname{Inf} \{\sigma(\tilde{V}, \pi) \mid (\tilde{V}, \pi) \in \mathscr{F}(V, M)\}$$

and if $\theta \in H_k(V; \mathbf{Z}/2\mathbf{Z})$ define

$$\sigma(\theta) = \operatorname{Inf}\left\{\sigma(M)\,\middle|\, \begin{array}{l} M \hookrightarrow V \times \mathbf{R}^s \text{ is an imbedding for some } s, \\ p_*[M] = \theta \text{ where } p \text{ is the projection} \end{array}\right\}$$

Then we have:

PROPOSITION 2.11 ([AK$_8$]). *If* $\theta \in H_k(V, \mathbf{Z}/2\mathbf{Z})$ *then* $\theta \in H_*^A(V; \mathbf{Z}/2\mathbf{Z})$ *if and only if* $\sigma(\theta) = 0$.

In particular this obstruction $\sigma(\theta)$ is a function of the codimension one obstruction of Theorem 2.6. It measures whether certain codimension one homology classes are transcendental. There is also a relative version of Nash's theorem:

THEOREM 2.12 ([AK$_3$]). *Let* M *be a closed smooth manifold and* $M_i \subset M$ $i = 0, \ldots, k$ *be closed smooth submanifolds in general position. Then there exists a nonsingular algebraic set* V *and a diffeomorphism.* $\lambda : M \to V$ *such that* $\lambda(M_i)$ *is a nonsingular algebraic subset of* V *for all* i.

A proof of special case: Here we give a proof of the case when each M_i is a codimension one submanifold. Since $\mathbf{RP}^n$ approximates $K(\mathbf{Z}/2\mathbf{Z}, 1)$ for n large, we can find imbeddings $\gamma_i : M \hookrightarrow \mathbf{RP}^n$ with $\gamma_i^{-1}(\mathbf{RP}^{n-1}) = M_i$. Consider the product imbedding $\gamma : M \hookrightarrow \prod_{i=1}^{k} \mathbf{RP}_i^n$, where $\mathbf{RP}_i^n = \mathbf{RP}^n$, $\gamma = (\gamma_1, \ldots, \gamma_k)$. Then by Theorem 2.8, after a small isotopy we can assume that $\gamma(M)$ is a nonsingular algebraic subset V of $\prod_{i=1}^{k} \mathbf{RP}_i^n \times \mathbf{R}^m$ for some m (since $\prod_{i=1}^{k} \mathbf{RP}_i^n$ has totally algebraic homology). Let $\pi_i : \prod_{i=1}^{k} \mathbf{RP}_i^n \times \mathbf{R}^m \to \mathbf{RP}^n$ be the projection to the i-th factor, and call $V_i = \pi_i^{-1}(\mathbf{RP}^{n-1}) \cap V$ then $V_i \approx M_i$ by transversality. In fact γ induces a diffeomorphism

$$(M; M_1, \ldots, M_k) \approx (V; V_1, \ldots, V_k)\,. \qquad \square$$

In [BT$_2$] another proof of this theorem is given. Theorem 2.12 can be used to produce distinct algebraic structures on smooth manifolds. If V is a smooth manifold we can define a usual structure set

$$\mathscr{S}_{\mathrm{Alg}}(V) = \left\{(V', g)\,\middle|\, \begin{array}{l} V' \text{ is a nonsingular algebraic set} \\ g : V' \to V \text{ is a diffeomorphism} \end{array}\right\}\Big/\sim$$

$\sim$ is the equivalence relation $(V', g) \sim (V'', h)$ if there is a birational diffeomorphism γ making the following commute

$$\begin{tikzcd}
V' \arrow[dr, "g"] \arrow[dd, "\gamma"'] & \\
 & V \\
V'' \arrow[ur, "h"'] &
\end{tikzcd}$$

$\mathcal{S}_{\mathrm{Alg}}(V)$ is the set of distinct algebraic structures on V. Hence a natural problem is to compute $\mathcal{S}_{\mathrm{Alg}}(V)$, or at least produce nontrivial elements of this set. For example if we take $M \subset V$ as in Proposition 2.10, then by Theorem 2.12 (V, M) is diffeomorphic to nonsingular algebraic sets (V', M'). Let $|V| = |V'|$ denote the underlying smooth structures and let $V \xrightarrow{g} |V|$, $V' \xrightarrow{g'} |V|$ be the forgetful maps. Then (V, g) and (V', g') are distinct elements of $\mathcal{S}_{\mathrm{Alg}}(|V|)$, otherwise M would be isotopic to a nonsingular algebraic subset of V.

An interesting question is whether algebraic structures on smooth manifolds satisfy the product structure theorem; that is, whether the natural map

$$\mathcal{S}_{\mathrm{Alg}}(M) \times \mathbf{R}^n \to \mathcal{S}_{\mathrm{Alg}}(M \times \mathbf{R}^n), \ (V, g) \mapsto (V \times \mathbf{R}^n, g \times id)$$

is surjection. The answer would be negative if one can find a smooth manifold M and $\theta \in H_*(M; \mathbf{Z}/2\mathbf{Z})$ such that M can not be diffeomorphic to a nonsingular algebraic set M' with $\theta \in H_*^A(M'; \mathbf{Z}/2\mathbf{Z})$. To see this, pick any smooth representative $N \xrightarrow{g} M$ of $\theta = g_*[N]$. By graphing g, we can assume $N \subset M \times \mathbf{R}^n$ for some n and g is induced by projection. By Theorem 2.12 we can find a diffeomorphism $\lambda : M \times \mathbf{R}^n \to V$ to a nonsingular algebraic set V with $\lambda(N)$ is an algebraic subset (one has to modify Theorem 2.12 to apply to this noncompact case). Then there can not exist a birational diffeomorphism $\mu : V \to M' \times \mathbf{R}^n$ where M' is a nonsingular algebraic set diffeomorphic to M, otherwise $\lambda(N) \xrightarrow{\mu} M' \times \mathbf{R}^n \xrightarrow{\text{projection}} M'$ would represent $\theta \in H_*^A(M'; \mathbf{Z}/2\mathbf{Z})$.

§3. BLOWING DOWN

Real algebraic sets obey some simple but useful topological properties:

PROPOSITION 3.1.

(a) *One point compactification an algebraic set is homeomorphic to an algebraic set.*

(b) *Given algebraic sets* $L \subset V$, *then* $V - L$ *is homeomorphic to an algebraic set.*

(c) *Given algebraic sets* $L \subset V$ *with* V *compact then* V/L *is homeomorphic to an algebraic set.*