

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 29 (1983)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE TOPOLOGY OF REAL ALGEBRAIC SETS
Autor: Akbulut, Selman / King, Henry
Kapitel: §2. Nonsingular Algebraic Sets
DOI: <https://doi.org/10.5169/seals-52981>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

§2. NONSINGULAR ALGEBRAIC SETS

The fact that closed smooth manifolds are diffeomorphic to nonsingular algebraic sets can be traced back to the following simple fact.

PROPOSITION 2.1. *Let L be a nonsingular algebraic set and K be a compact set with $L \subset K \subset \mathbf{R}^n$, let $f: \mathbf{R}^n \rightarrow \mathbf{R}$ be a smooth function with $f|_L = u$ for some entire rational function u . Then there is an entire rational function $p: \mathbf{R}^n \rightarrow \mathbf{R}$ which approximates f arbitrarily closely near K with $p|_L = u$ (if u is a polynomial then p can be taken to be a polynomial). Furthermore if $f - u$ has compact support then p can approximate f on all of \mathbf{R}^n .*

Proof: First write $f - u = \sum_i a_i \cdot \beta_i$ where a_i are smooth functions and $\beta_i \in I(L)$. Clearly we can do this locally, and then by putting these local expressions together by partitions of unity we get the global expression. We approximate $a_i(x)$ by polynomials $\alpha_i(x)$ near K and let $p = u + \sum_i \alpha_i \cdot \beta_i$. $p(x)$ has the required properties. If $p - u$ has compact support we can define a smooth function $g: S^n \rightarrow \mathbf{R}$ by $g = (f - u) \circ \theta$ on $S^n - (0, 1)$ and $g(0, 1) = 0$, where $S^n \subset \mathbf{R}^n \times \mathbf{R}$ is the unit sphere and $\theta: S^n - (0, 1) \rightarrow \mathbf{R}^n$ is the stereographic projection, $\theta(x, t) = \frac{x}{1 - t}$. Then

$$g: (S^n, \theta^{-1}(L) \cup (0, 1)) \rightarrow (\mathbf{R}, 0)$$

hence by the first part of the theorem g can be approximated by an entire rational function

$$\hat{p}: (S^n, \theta^{-1}(L) \cup (0, 1)) \rightarrow (\mathbf{R}, 0).$$

Let $p = \hat{p} \circ \theta^{-1} + u$. □

The following was introduced in [AK₂] to simplify Nash's and Tognoli's theorems.

PROPOSITION 2.2 (Normalization). *Given $L \subset K \subset \mathbf{R}^n$, $W \subset \mathbf{R}^m$ where L, W are nonsingular algebraic sets and K is a compact set, and $f: K \rightarrow W$ a smooth function with $f|_L = u$ for some entire rational function $u: L \rightarrow W$. Then there is an algebraic set $Z \subset \mathbf{R}^n \times \mathbf{R}^m$ and an entire rational function*

$p : Z \rightarrow W$ and an open neighborhood U of K in \mathbf{R}^n and a smooth function $\varphi : (U, L) \rightarrow (\mathbf{R}^m, 0)$ such that

- (i) The set $\tilde{U} = \{(x, \varphi(x)) \mid x \in U\} \subset \mathbf{R}^n \times \mathbf{R}^m$ is an open nonsingular subset of Z .
- (ii) p is arbitrarily close to $f \circ \pi$ on \tilde{U} where π is the projection to the first factor.
- (iii) $L \times 0 \subset \tilde{U}$ and $p|_{L \times 0} = u$.

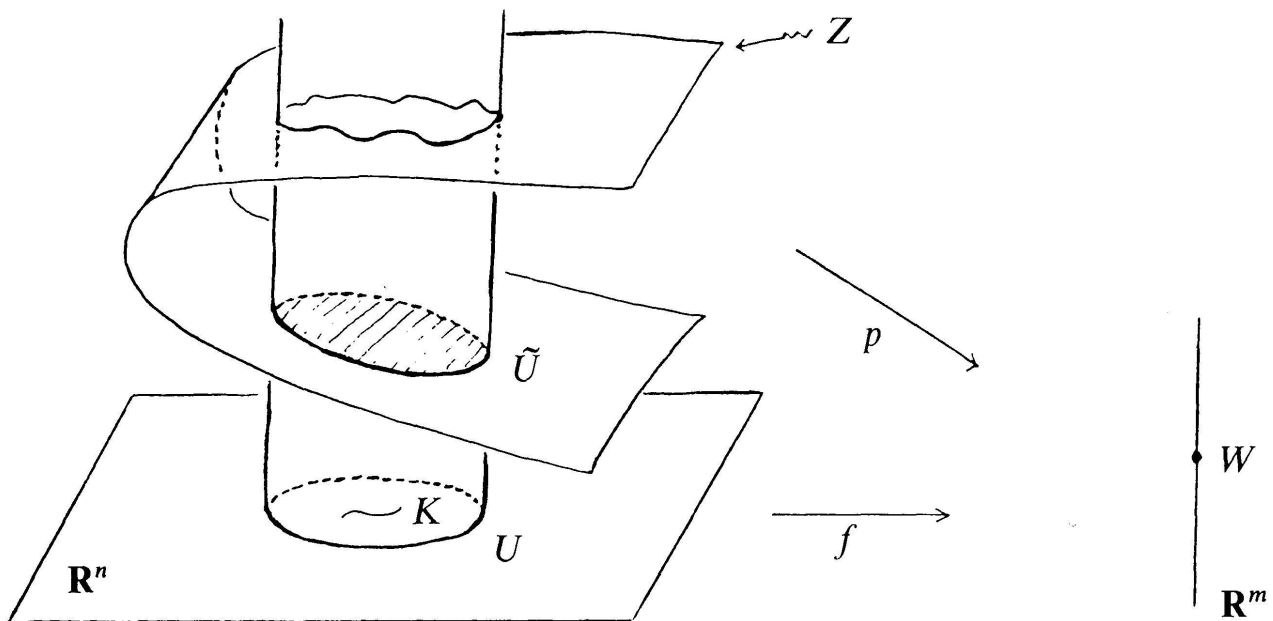
Proof: Let $\delta : \mathbf{R}^m \rightarrow \mathbf{R}^{m^2}$ be an entire rational function with

$$\delta(x) \in G(m, m - \dim W)$$

is the normal plane to W at $x \in W$ (from §0). By Proposition 2.1 there is an entire rational function $g : \mathbf{R}^n \rightarrow \mathbf{R}^m$ which approximates f on K with $g|_L = u$. Define:

$$Z = \{(x, y) \in \mathbf{R}^n \times \mathbf{R}^m \mid g(x) + y \in W, \delta(g(x) + y)y = y\}$$

$$p : Z \rightarrow \mathbf{R}^m, p(x, y) = g(x) + y$$



Clearly Z is an algebraic set. Let U be a small open tubular neighborhood of K such that g is arbitrarily close to f on U . Therefore when $x \in U$ there is a unique closest point $v(x)$ on W to $g(x)$. Define $\varphi(x) = v(x) - g(x)$ to be the vector from $g(x)$ to $v(x)$. Hence $\varphi(x)$ is perpendicular to W at $v(x) = g(x) + \varphi(x)$, so $\varphi(x)$ is the unique "small" solution of the equations

$$\left\{ \begin{array}{l} g(x) + y \in W \\ \delta(g(x) + y)y = y \end{array} \right\} \quad \text{which is} \quad \left\{ \begin{array}{l} g(x) + y \in W \\ y \text{ is } \perp \text{ to } W \text{ at } g(x) + y \end{array} \right\}$$

Hence $\tilde{U} = \{(x, \varphi(x)) \mid x \in U\}$ has the property

$$\tilde{U} = Z \cap U \times \{y \in \mathbf{R}^m \mid |y| < \varepsilon\}$$

for some small $\varepsilon > 0$. Clearly Z, U, p has the required properties. \square

THEOREM 2.3 (Generalized Seifert Theorem). *Let $M^m \subset V^v$ be a closed smooth submanifold of a nonsingular algebraic set V , imbedded with a trivial normal bundle, and let $L \subset M$ be a nonsingular algebraic set. Then by an arbitrarily small isotopy M is isotopic to a component of a nonsingular algebraic subset of V fixing L .*

Proof: Let $V \subset \mathbf{R}^n$ and let W, U be small open neighborhoods of M^m in V^v , and in \mathbf{R}^n respectively. Let $f : W \rightarrow \mathbf{R}^{v-m}$ be the trivialization map of the normal bundle of M in V , f is transverse to $0 \in \mathbf{R}^{v-m}$ and $f^{-1}(0) = M$. Then extend f to $f : U \rightarrow \mathbf{R}^{v-m}$. Since $f|_L = 0$ by Proposition 2.1 we can approximate f on $\text{Closure}(U)$ by a polynomial $F : (\mathbf{R}^n, L) \rightarrow (\mathbf{R}^{v-m}, 0)$. By transversality $F^{-1}(0) \cap W$ is isotopic to $f^{-1}(0) \cap W = M$. In general $F^{-1}(0)$ might have extra components outside of U . \square

It is interesting to note that in general the extra components of $F^{-1}(0)$ can not be removed, there are homotopy theoretical obstructions $[AK_8]$ (even when $L = \emptyset$).

Remark 2.4. In Theorem 2.3 it is not necessary to assume that L is nonsingular, it suffices to assume that some open neighborhood W of L in M coincides with an open subset of a nonsingular algebraic set. The proof is the same except it requires a slight modification in Proposition 2.1 (see $[AK_2]$).

THEOREM 2.5 (Generalized Nash theorem). *Let $M^m \subset \mathbf{R}^n$ be a closed smooth submanifold, and $L \subset M$ be a nonsingular algebraic set. Assume that some open neighborhood W of L in M is an open subset of some nonsingular algebraic set. Then by an arbitrarily small isotopy M can be isotoped to a nonsingular component of an algebraic subset of $\mathbf{R}^n \times \mathbf{R}^s$ keeping L fixed (for some s).*

Proof: Let U be an open tubular neighborhood of M in \mathbf{R}^n and $f : U \rightarrow E(n, k)$ be the map which classifies the normal bundle of M in U . $f \pitchfork G(n, k)$ and $f^{-1}(G(n, k)) = M$. By using W we can assume $f|_L = u$ for some entire rational function u (see §0). By Proposition 2.2 there is a nonsingular open subset \tilde{U} of an algebraic set $Z \subset \mathbf{R}^n \times \mathbf{R}^s$ for some s , and an entire rational function $p : \tilde{U} \rightarrow E(n, k)$ which makes the following commute

$$\begin{array}{ccc}
 \mathbf{R}^n \times \mathbf{R}^s \supset \tilde{U} & & \\
 \downarrow \pi & \searrow p & \\
 \mathbf{R}^n \supset U & \xrightarrow{f} & E(n, k) \supset G(n, k)
 \end{array}$$

where π is projection, and $f \circ \pi$ is close to p , and $L \times 0 \subset \tilde{U}$ with $p|_{L \times 0} = u$.

$$\tilde{U} = \{(x, \varphi(x)) \mid x \in U\}$$

for some smooth function $\varphi(x)$. Let $\hat{p}(x) = p(x, \varphi(x))$ then \hat{p} is close to f on U . By transversality $\hat{p}^{-1}(G(n, k)) \cap U$ is isotopic to $f^{-1}(G(n, k)) \cap U = M$ in U . Since π is an isomorphism on \tilde{U} and $p = \hat{p} \circ \pi$,

$$p^{-1}(G(n, k)) \cap \tilde{U} = \pi^{-1}(\hat{p}^{-1}(G(n, k)) \cap U) \approx M.$$

$p^{-1}(G(n, k)) \cap \tilde{U}$ is a component of an algebraic set by construction and nonsingular by transversality, furthermore it contains $L \times 0$. \square

Let V be a nonsingular real algebraic set of dimension n . Recall $AH_{n-1}(V; \mathbf{Z}/2\mathbf{Z})$ is the subgroup of $H_{n-1}(V; \mathbf{Z}/2\mathbf{Z})$ generated by nonsingular algebraic subsets. We define

$$H_{n-1}^t(V) = H_{n-1}(V; \mathbf{Z}/2\mathbf{Z}) / AH_{n-1}(V; \mathbf{Z}/2\mathbf{Z}),$$

which we call the group of *codimension one transcendental cycles*. For any codimension and closed smooth submanifold $M \subset V$ let $\alpha(M)$ be the image of the fundamental homology class $[M]$ under the quotient map.

THEOREM 2.6 ([AK₈]). *Any codimension one closed smooth submanifold $M \subset V$ of a nonsingular algebraic set V is isotopic to a nonsingular algebraic subset by an arbitrarily small isotopy if and only if $\alpha(M) = 0$.*

Sketch of proof: For simplicity assume that M has a trivial normal bundle and $[M]$ is represented by a single nonsingular algebraic subset W of V . If $M \cap W = \emptyset$ then $M \cup W$ separates V into two components V_+ , V_- with one of them, say V_+ , is compact (since M is homologous to W). Let $f: (V, M \cup W) \rightarrow (\mathbf{R}, 0)$ be a smooth function with $f > 0$ on V_+ and $f < 0$ on V_- . We can assume that f is transversal to 0 and is constant outside of a compact set containing V_+ . By Proposition 2.1 we can approximate f by a polynomial $F: (V, W) \rightarrow (\mathbf{R}, 0)$, then by transversality $F^{-1}(0) = M' \cup W$ where M' is isotopic to M . $M' \cup W$ is a nonsingular algebraic set hence M' is a nonsingular algebraic set.

If $M \cap W \neq \emptyset$ then we can find a smooth representative N of $[M]$ with $N \cap M = \emptyset$ and $N \cap W = \emptyset$. By the first part we can isotope N to a nonsingular algebraic set N' by a small isotopy. Hence $N' \cap M = \emptyset$; and since N' is homologous to M by the previous case M is isotopic to a nonsingular algebraic set by a small isotopy.

The proof of the case M does not have a trivial normal bundle is more difficult, we refer the reader to [AK₈]. \square

Proposition 2.10 implies that $H_{n-1}^t(V)$ is nontrivial in general. One of the corollaries of Theorem 2.6 is that codimension one nonsingular algebraic sets can be moved around by isotopies. A natural generalization of this fact is:

THEOREM 2.7 (Algebraic transversality [AK₈]). *Let V be a nonsingular algebraic set and $M \subset V$ be a stable algebraic subset. Let N be a smooth subcomplex of V . Then there exists an arbitrarily small isotopy $f_t: M \rightarrow V$ with $f_0(M) = M$ and $f_1(M)$ is a stable algebraic subset transverse to N .*

Let $\eta_*(V)$ be the unoriented bordism group of a nonsingular algebraic set V . Let $\eta_*^A(V)$ be the subgroup of $\eta_*(V)$ generated by entire rational maps $f: M \rightarrow V$ where M is a compact nonsingular algebraic set. By taking graph of f one easily sees that every element of $\eta_*^A(V)$ has a representative (M, f) , where $M \subset V \times \mathbf{R}^n$ is a nonsingular algebraic set for some n , and f is induced by projection.

THEOREM 2.8. *Let $f: M \rightarrow V$ be a map from a closed smooth manifold to a nonsingular algebraic set V . Then $(M, f) \in \eta_*^A(V)$ if and only if $f \times 0$ can be approximated by an imbedding onto a nonsingular algebraic subset of $V \times \mathbf{R}^n$ for some n .*

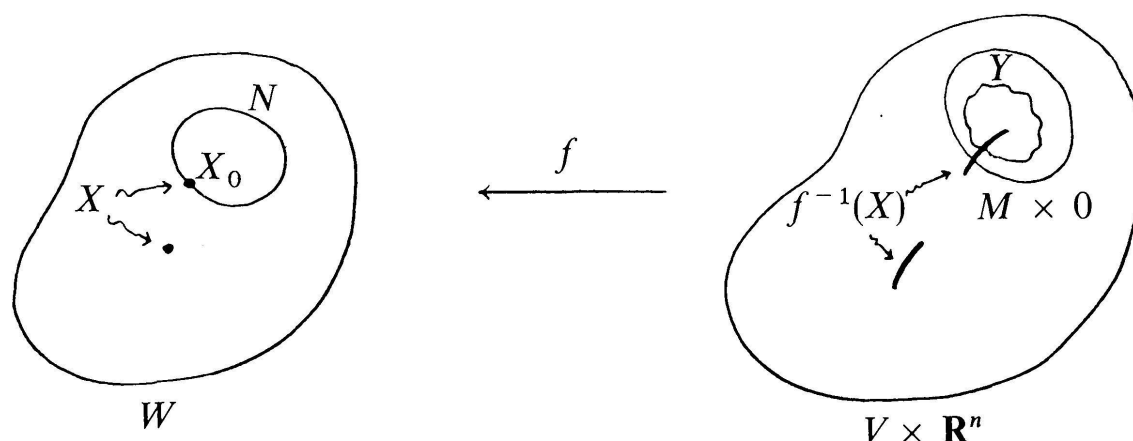
Proof: One way the proof is trivial. Assume $(M, f) \in \eta_*^A(V)$, then there is a smooth manifold Z and a map $F: Z \rightarrow V$ with $\partial Z = M \cup N$ and N is a nonsingular algebraic set, $F|_M = f$ and $F|_N$ is an entire rational function. Let \hat{Z} be the double of Z i.e. $\hat{Z} = \partial(Z \times [-1, 1])$. By taking graph of F we may assume $Z \subset V \times \mathbf{R}^s$ is imbedded for some s . In particular $N \subset Z$ is a nonsingular algebraic subset of $V \times \mathbf{R}^s$. Then extend this imbedding to an imbedding $\hat{Z} \subset V \times \mathbf{R}^s \times \mathbf{R}$ which is identity on $N \times (-1, 1)$. Then by Theorem 2.5 we can isotope \hat{Z} to a nonsingular component of an algebraic set $Y \subset V \times \mathbf{R}^n$ for some n with $N \subset Y$. Since the codimension one submanifolds N and M of \hat{Z} are homologous, M can be isotoped to a nonsingular algebraic subset of Y , by Theorem 2.6. \square

COROLLARY 2.9 (Tognoli [To]). *Every closed smooth manifold is diffeomorphic to a nonsingular algebraic set.*

The hypothesis of Theorem 2.8 is not void in fact we have:

PROPOSITION 2.10 ([AK₈]). *For any k there exist a nonsingular connected algebraic set V and a closed smooth codimension k submanifold $M \subset V$ which can not be isotopic to a nonsingular algebraic subset in $V \times \mathbf{R}^n$ for any n .*

Proof: Let $W = \mathbf{R}^m$ with $m - k$ even, and X be an algebraic subset given by $x_2^4 + (x_1^2 - 1) \cdot (x_1^2 - 4) = 0$ and $x_3 = x_4 = \dots = x_m = 0$. X is a nonsingular irreducible algebraic set of two components $X_0 \cup X_1$ each of which is homeomorphic to a circle. Let N be any smooth submanifold of W with $N \cap X = X_0$, and $\dim(N) = m - k$. Then let $M = B(N, X_0)$, $V = B(W, X) \xrightarrow{\pi} W$ be topological and algebraic blowups, respectively. Assume that $M \times 0$ was isotopic to an algebraic subset Y of $V \times \mathbf{R}^n$ by a small isotopy. Then we get a compact nonsingular algebraic set $Z = Y \cap (\pi \circ p)^{-1}(X)$ and an entire rational function $f = \pi \circ p$ where $p: V \times \mathbf{R}^n \rightarrow V$ is the projection. Furthermore $f: Z \rightarrow \mathbf{R}^m$ has the properties: $f(Z) = X_0$ and $f^{-1}(x) \approx \mathbf{RP}^{m-k-2}$ for $x \in X_0$ by transversality. Hence since $\bar{X}_0 = X$ and $\chi(\mathbf{RP}^{m-k-2})$ is odd we get a contradiction to Lemma 0.2. \square



Recall $\eta_*(V) \approx H_*(V; \mathbf{Z}/2\mathbf{Z}) \otimes \eta_*(\text{point})$ and $\eta_*(V)$ is generated by $Q \times N \xrightarrow{\pi} Q \xrightarrow{g} V$ where π is the projection and N is a generator of $\eta_*(\text{point})$ and $g_*[Q]$ is a generator of $H_*(V; \mathbf{Z}/2\mathbf{Z})$. Given $(M, f) \in \eta_*(V)$ with $(M, f) = \sum \theta_i \otimes U_i$ then it follows that $(M, f) \in \eta_*^4(V)$ if each $\theta_i \in H_*^4(V; \mathbf{Z}/2\mathbf{Z})$ ([AK₂]). If an algebraic set V has the property $H_*(V; \mathbf{Z}/2\mathbf{Z}) = H_*^4(V; \mathbf{Z}/2\mathbf{Z})$ for all $*$ we say that V has *totally algebraic homology*; therefore such algebraic sets have the

property $\eta_*(V) = \eta_*^A(V)$. \mathbf{RP}^m and more generally $G(n, m)$ are examples of algebraic sets with totally algebraic homology, because their homology is generated by Schubert cycles. This property is invariant under cross products. Also if $L \subset V$ are nonsingular algebraic sets with totally algebraic homology, then so is $B(V, L)$ (Proposition 6.1 of [AK₆]). It is still an open question that whether any closed smooth manifold is diffeomorphic to a nonsingular algebraic set with totally algebraic homology.

Therefore it would be useful to understand when a given homology class $\theta \in H_*(V; \mathbf{Z}/2\mathbf{Z})$ of a nonsingular algebraic set V lies in $H_*^A(V; \mathbf{Z}/2\mathbf{Z})$. This can be detected by a single obstruction $\sigma(\theta)$ as follows. Let $M \subset V$ be a fine submanifold of a nonsingular algebraic set, in particular

$$M = V_0 \subset V_1 \subset \dots \subset V_r \subset V_{r+1} = V$$

for some closed smooth manifolds $\{V_i\}$ with $\dim(V_{i+1}) = \dim(V_i) + 1$, then let

$$\tilde{\alpha}(M) = \text{Inf} \{k \mid \alpha(V_i) = 0 \text{ for } i \geq k\}$$

(make the convention $\alpha(V_{r+1}) = 0$). Recall the definition of $\alpha(V_r) \in H_{n-1}^t(V)$, where $n = \dim(V)$. Theorem 2.6 says that if $\alpha(V_r) = 0$ then V_r can be made a nonsingular algebraic subset of V and therefore $\alpha(V_{r-1}) \in H_{n-2}^t(V_r)$ is defined... etc. Hence by continuing this fashion we see that if $\tilde{\alpha}(M) = 0$ then M is isotopic to an algebraic subset of V .

If $M \subset V$ is just a smooth submanifold of V , then let $\mathcal{F}(V, M)$ be the set of all fine topological multiblowups $\tilde{V} \xrightarrow{\pi} V$ along M ($\mathcal{F}(V, M) \neq \emptyset$ by Theorem 1.2 and the remarks proceeding it):

$$\tilde{V} = V_k \xrightarrow{\pi_k} V_{k-1} \xrightarrow{\pi_{k-1}} \dots \xrightarrow{\pi_1} V_0 = V,$$

where $V_i = B(V_{i-1}, L_{i-1})$, and $L_i \subset V_i$, $\tilde{M} \subset V_k$ are all fine submanifolds. Make the convention $\tilde{M} = L_k$ then for $(\tilde{V}, \pi) \in \mathcal{F}(V, M)$ define

$$\sigma(\tilde{V}, \pi) = \text{Inf} \{k - n \mid \tilde{\alpha}(L_i) = 0 \text{ for } i \leq n\}$$

Then $\sigma(\tilde{V}, \pi) = 0$ implies that all $\tilde{\alpha}(L_i) = 0$, hence inductively we can assume that $L_i \subset V_i$ are nonsingular algebraic subsets and therefore we can make $\tilde{V} \xrightarrow{\pi} V$ an algebraic multiblowup and $\tilde{M} \subset \tilde{V}$ an algebraic subset. In fact $\sigma(\tilde{V}, \pi) = 0$ if and only if $\tilde{V} \xrightarrow{\pi} V$ is a stable algebraic multiblowup along M . Let

$$\sigma(M) = \text{Inf} \{\sigma(\tilde{V}, \pi) \mid (\tilde{V}, \pi) \in \mathcal{F}(V, M)\}$$

and if $\theta \in H_k(V; \mathbf{Z}/2\mathbf{Z})$ define

$$\sigma(\theta) = \text{Inf} \left\{ \sigma(M) \left| \begin{array}{l} M \hookrightarrow V \times \mathbf{R}^s \text{ is an imbedding for some } s, \\ p_*[M] = \theta \text{ where } p \text{ is the projection} \end{array} \right. \right\}$$

Then we have:

PROPOSITION 2.11 ([AK₈]). *If $\theta \in H_k(V, \mathbf{Z}/2\mathbf{Z})$ then $\theta \in H_*^A(V; \mathbf{Z}/2\mathbf{Z})$ if and only if $\sigma(\theta) = 0$.*

In particular this obstruction $\sigma(\theta)$ is a function of the codimension one obstruction of Theorem 2.6. It measures whether certain codimension one homology classes are transcendental. There is also a relative version of Nash's theorem:

THEOREM 2.12 ([AK₃]). *Let M be a closed smooth manifold and $M_i \subset M$ $i = 0, \dots, k$ be closed smooth submanifolds in general position. Then there exists a nonsingular algebraic set V and a diffeomorphism $\lambda: M \rightarrow V$ such that $\lambda(M_i)$ is a nonsingular algebraic subset of V for all i .*

A proof of special case: Here we give a proof of the case when each M_i is a codimension one submanifold. Since \mathbf{RP}^n approximates $K(\mathbf{Z}/2\mathbf{Z}, 1)$ for n large, we can find imbeddings $\gamma_i: M \hookrightarrow \mathbf{RP}^n$ with $\gamma_i^{-1}(\mathbf{RP}^{n-1}) = M_i$. Consider the product imbedding $\gamma: M \hookrightarrow \prod_{i=1}^k \mathbf{RP}_i^n$, where $\mathbf{RP}_i^n = \mathbf{RP}^n$, $\gamma = (\gamma_1, \dots, \gamma_k)$. Then by Theorem 2.8, after a small isotopy we can assume that $\gamma(M)$ is a nonsingular algebraic subset V of $\prod_{i=1}^k \mathbf{RP}_i^n \times \mathbf{R}^m$ for some m (since $\prod_{i=1}^k \mathbf{RP}_i^n$ has totally algebraic homology). Let $\pi_i: \prod_{i=1}^k \mathbf{RP}_i^n \times \mathbf{R}^m \rightarrow \mathbf{RP}^n$ be the projection to the i -th factor, and call $V_i = \pi_i^{-1}(\mathbf{RP}^{n-1}) \cap V$ then $V_i \approx M_i$ by transversality. In fact γ induces a diffeomorphism

$$(M; M_1, \dots, M_k) \approx (V; V_1, \dots, V_k).$$

□

In [BT₂] another proof of this theorem is given. Theorem 2.12 can be used to produce distinct algebraic structures on smooth manifolds. If V is a smooth manifold we can define a usual structure set

$$\mathcal{S}_{\text{Alg}}(V) = \left\{ (V', g) \left| \begin{array}{l} V' \text{ is a nonsingular algebraic set} \\ g: V' \rightarrow V \text{ is a diffeomorphism} \end{array} \right. \right\} / \sim$$

\sim is the equivalence relation $(V', g) \sim (V'', h)$ if there is a birational diffeomorphism γ making the following commute

$$\begin{array}{ccc}
 V' & \xrightarrow{g} & V \\
 \gamma \downarrow & & \uparrow h \\
 V'' & \xrightarrow{h} & V
 \end{array}$$

$\mathcal{S}_{\text{Alg}}(V)$ is the set of distinct algebraic structures on V . Hence a natural problem is to compute $\mathcal{S}_{\text{Alg}}(V)$, or at least produce nontrivial elements of this set. For example if we take $M \subset V$ as in Proposition 2.10, then by Theorem 2.12 (V, M) is diffeomorphic to nonsingular algebraic sets (V', M') . Let $|V| = |V'|$ denote the underlying smooth structures and let $V \xrightarrow{g} |V|$, $V' \xrightarrow{g'} |V|$ be the forgetful maps. Then (V, g) and (V', g') are distinct elements of $\mathcal{S}_{\text{Alg}}(|V|)$, otherwise M would be isotopic to a nonsingular algebraic subset of V .

An interesting question is whether algebraic structures on smooth manifolds satisfy the product structure theorem; that is, whether the natural map

$$\mathcal{S}_{\text{Alg}}(M) \times \mathbf{R}^n \rightarrow \mathcal{S}_{\text{Alg}}(M \times \mathbf{R}^n), (V, g) \mapsto (V \times \mathbf{R}^n, g \times \text{id})$$

is surjection. The answer would be negative if one can find a smooth manifold M and $\theta \in H_*(M; \mathbf{Z}/2\mathbf{Z})$ such that M can not be diffeomorphic to a nonsingular algebraic set M' with $\theta \in H_*^A(M'; \mathbf{Z}/2\mathbf{Z})$. To see this, pick any smooth representative $N \xrightarrow{g} M$ of $\theta = g_*[N]$. By graphing g , we can assume $N \subset M \times \mathbf{R}^n$ for some n and g is induced by projection. By Theorem 2.12 we can find a diffeomorphism $\lambda: M \times \mathbf{R}^n \rightarrow V$ to a nonsingular algebraic set V with $\lambda(N)$ is an algebraic subset (one has to modify Theorem 2.12 to apply to this noncompact case). Then there can not exist a birational diffeomorphism $\mu: V \rightarrow M' \times \mathbf{R}^n$ where M' is a nonsingular algebraic set diffeomorphic to M , otherwise $\lambda(N) \xrightarrow{\mu} M' \times \mathbf{R}^n \xrightarrow{\text{projection}} M'$ would represent $\theta \in H_*^A(M'; \mathbf{Z}/2\mathbf{Z})$.

§3. BLOWING DOWN

Real algebraic sets obey some simple but useful topological properties:

PROPOSITION 3.1.

- (a) One point compactification an algebraic set is homeomorphic to an algebraic set.
- (b) Given algebraic sets $L \subset V$, then $V - L$ is homeomorphic to an algebraic set.
- (c) Given algebraic sets $L \subset V$ with V compact then V/L is homeomorphic to an algebraic set.