§4. Local knot theory in brief

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interior of a ball or a bidisk), well-behaved at the boundary; a knot-theorist can study either of two codimension-2 situations—the complex curve in its ambient space, or the boundary of this pair.

This middle panel of the triptych has been less studied than the other two, though it is of obvious relevance to both.

§3. Résumé of basic definitions

By *complex surface* I mean a smooth manifold of 4 real dimensions, equipped with a complex structure. A *complex curve* Γ in a complex surface M is a closed subset which is locally of the form $\{(z, w) \in U \subset \mathbf{C}^2 : f(z, w) = 0\}$ where $f : U \to \mathbf{C}$ is a nonconstant complex analytic function. A *Riemann surface* is a smooth manifold of 2 real dimensions, equipped with a complex structure.

It is a fundamental fact, to which is due the especial appositeness of classical knot theory to the study of curves in surfaces, that any complex curve $\Gamma \subset M$ has a *resolution* of the following sort: There is a Riemann surface R, and a holomorphic mapping $r : R \to M$, so that $r(R) = \Gamma$; in fact, there is a discrete (possibly empty) subset $\mathscr{S}(\Gamma) \subset \Gamma$, the *singular locus of* Γ *in* M, so that the *regular locus* $\mathscr{R}(\Gamma) = \Gamma - \mathscr{S}(\Gamma)$ is a Riemann surface, and R is the union (with what turns out to be a unique topology and complex structure) of $\mathscr{R}(\Gamma)$, on which r is the identity, and a discrete set $r^{-1}(\mathscr{S}(\Gamma)) \subset R$ mapping finitely-to-one onto $\mathscr{S}(\Gamma)$.

The singular locus is, of course, exactly the set of points of Γ at which, no matter what the local representation of Γ as the zeroes of an analytic function $f(z, w)$, the (complex) gradient vector ∇f vanishes.

If P is a point of Γ, and $Q \in r^{-1}(P) \subset R$, then the germ at P of the r-image of a small disk on R centered at Q is called a *branch* of Γ at P. (Abusively, "branch" may also be used below to refer to some representatives of this germ.) Naturally, at a regular point there is only one branch; but there may be only one branch at a point, and the point still be singular.

References: [G-R], [Mi 2].

§4. Local knot theory in brief

Using local coordinates in the resolution R and the ambient surface M, one sees that each branch of a curve Γ can be parametrized either by $z = t, w = 0$ or (more interestingly) by some pair $z = t^m, \quad w = t^n + c_{n+1}t^{n+1} + \ldots$

$+ c_N t^N$, $t \in \mathbf{C}$, with $n > m$. (In the original choice of coordinates, r might well involve genuine power series; but it is not hard to make a formal change of coordinates to one of the forms above, involving only polynomials; and it is not much harder to prove a comparison theorem, the remote ancestor of that of M. Artin, which shows that actually the formal change of coordinates can be taken to be somewhere convergent.) Consider the "approximations" to such a branch, gotten by dropping all terms of w from some degree on up: so the first approximation is (t^m, t^n), and the $(N-n+1)$*st* is the branch we began with. Each of these is itself a map onto a branch of some curve; generally not one-to-one.

Define integers $g(1), \ldots, g(N-n+1)$ by saying that the k-th approximation is $g(k)$-to-one in a punctured neighborhood of $t = 0$. Then $g(1) = GCD(m, n)$, $g(k+1)$ divides $g(k)$, and $g(N-n+1) = 1$. These integers can be calculated as follows. Let $\mathbf{C}[[t]]$ be the algebra of formal power series, with unique maximal ideal $m = t\mathbf{C}[[t]]$. Let A_k be the m-adically closed subalgebra generated by 1 and the components of the k-th approximation. Then $g(k)$ is the least integer g such that $A_k \subset \mathbf{C}[[t^g]] \subset \mathbf{C}[[t]]$. (One gets the same answer starting from the algebra $\mathbf{C}\{t\}$ of somewhere-convergent power series.) A parametrization of the branch covered by the k-th approximation is $z = t^{m/g(k)}$, $w = t^{n/g(k)} + \ldots + c_{n+k-1} t^{(n+k)/g(k)}$.

The knots in which we are interested arise when we intersect the branch under investigation with the boundary of an infinitesimal 4-disk containing the singular point. The 4-disk used may be either a *round disk* $D_\varepsilon^4 = \{(z, w) : |z|^2 + |w|^2 = \varepsilon^2\}$ with boundary the *round sphere* S_ε^3, or a *bidisk* $D(\varepsilon_1, \varepsilon_2) = \{(z, w) : |z| \leqslant \varepsilon_1, |w| \leqslant \varepsilon_2\}$, with boundary comprised of two solid tori $\partial_1 D(\varepsilon_1, \varepsilon_2) = \{|z| = \varepsilon_1, |w| \leqslant \varepsilon_2\}$ and $\partial_2 D(\varepsilon_1, \varepsilon_2) = \{|z| \leqslant \varepsilon_1, |w| = \varepsilon_2\}$ which together make up a 3-sphere with corners. Whether one uses round disks or bidisks, one obtains a knot of the same type. The bidisk boundary is more convenient here, when we are studying the branch parametrically; from the assumption that $n > m$ we can see that, for sufficiently small $\varepsilon > 0$, the branch intersects $\partial D(\varepsilon, \varepsilon)$ only along $\partial_1 D(\varepsilon, \varepsilon)$.

The first approximation to the branch actually meets $\partial_1 D$ on the torus $\{|z| = \varepsilon, |w| = \varepsilon^{n/m}\}$, where it covers, $g(1)$ to one, a *torus knot* of type $O\{m/g(1), n/g(1)\}$. (Here is the notation I am using, cf. [Ru 4]. If K is any oriented knot in an oriented 3-sphere, with closed tubular neighborhood $N(K)$, let L be an oriented simple closed curve on $\partial N(K)$ which is not null-homologous on this torus; then there are relatively prime integers p and q so that L has linking number q with K and represents p times the class of K in $H_1(N(K); \mathbf{Z})$. We then call L a *cable of type* (p, q) *about* K and denote it by $K\{p, q\}$. When cabling is iterated, excess curly braces become semicolons. The unknot is denoted by O; a cable about the unknot is also called a *torus knot*; a cable about ... a cable about the unknot is an *iterated torus knot*.) This knot type does not change when ε is made smaller.

Now suppose that for all sufficiently small $\varepsilon > 0$, the k-th approximation to a branch intersects $\partial D(\varepsilon, \varepsilon)$ in a knot of type $O\{p_1, q_1; \ldots; p_k, q_k\}$. Considering how we pass to the next approximation we see that there are relatively prime integers p_{k+1} and q_{k+1} so that, for all sufficiently small $\varepsilon > 0$, the $(k+1)$-st approximation to the branch intersects $\partial D(\varepsilon, \varepsilon)$ in a knot of type $O\{p_1, q_1; \ldots; p_k, q_k; p_{k+1}, q_{k+1}\}$. (The difference between successive approximations is 0 or a monomial $c_{n+k}t^{n+k} \not\equiv 0$, which contributes an "epicycle" that for small enough ε precisely creates a cabling.) In fact, $p_{k+1} = g(k)/g(k+1)$ (note that for any K and q, $K\{1, q\}$ is the same knot type as K); the formula for q_{k+1} is more complicated, and we won't give it.

Consider a curve with a singular point at which there are two or more branches. Coordinates in the ambient surface can be chosen so that each branch differs only by a diagonal linear transformation in (z, w) from one of the form just studied (including the non-singular case $z = t, w = 0$). Each branch individually contributes an iterated torus knot to the *link of the singularity*, $\Gamma \cap \partial D(\varepsilon, \varepsilon)$; and in fact they all fit together nicely. An elegant description of how they do is given in [E-N]; see also, and for this section generally, [Lê] and [Mi 2] and references cited therein.

After Burau, Zariski, *et al.*, had established that any point of a curve in a (non-singular) surface had local topology that was completely described by this link-type invariant, the strictly topological investigation of singular points seems to have languished for some decades. (The algebraic geometers, of course, had also established that this link-type invariant—more precisely, the sequences of pairs (p_i, q_i) for each branch, and the linking numbers between the iterated torus knots of different branches, from which the whole link of the singularity can be reconstructed—was equivalent to some numerical invariants which had long been known and which could be detected purely algebraically, namely, the Puiseux pairs of the various branches and the intersection multiplicity of the pairs of branches. They also pressed forward with their investigations of continuous invariants within the family of singularities of a given link type. But that is another story.) However, in the late 1960's, Milnor [Mi 2] gave new life to the subject when he showed that the link of a singularity was a "fibred", or Neuwirth-Stallings, link.

Milnor's proof uses the round-sphere model. He shows that, if $\Gamma \subset \mathbf{C}^2$ is the zero-locus of $p(z, w) \in \mathbf{C}[z, w]$, $p(0, 0) = 0$, then for all sufficiently small $\varepsilon > 0$, the restriction ϕ of the map $\arg p : \mathbf{C}^2 - \Gamma \to S^1 : (z, w) \mapsto p(z, w)/|p(z, w)|$ to

$S_\varepsilon^3 - \Gamma$ is the projection map of a fibration over S^1. The fibre is diffeomorphic to the interior of the surface $F_0 = S_\varepsilon^3 \cap \{(z, w) : p(z, w)$ is real and non-negative$\}$. (Note that the change in viewpoint from bidisk boundary to round sphere is accompanied by a change from branch-as-parametrized-disk to branch-as-level-set.)

We will see below that the link of a singularity is in a natural way a closed strictly positive braid; I will give a geometric proof of the well-known fact that such a closed braid is a fibred link.

Inspired by Milnor's Fibration Theorem, a number of mathematicians began investigations of knot-theoretical properties of the links of singularities. The fibration ϕ determines an autodiffeomorphism of F_0 (fixed on the boundary), unique up to isotopy relative to the boundary, which is variously called the *characteristic map*, *holonomy*, or *monodromy* of the fibration; it induces an automorphism (also called the monodromy) of the integral homology of F_0. From the homology monodromy one can calculate the Alexander polynomial of the link of the singularity; this was done in [Lê], where it was also shown that two branches defined iterated torus knots in the same knot-cobordism class if and only if they defined knots of the same knot type, the proof following from a study of the roots of the Alexander polynomials.

I wondered how independent these distinct knot-cobordism classes might be, in the knot-cobordism group; in particular, I asked [Ru 6] whether the equation $[K_0] = \sum_{i=1}^{n} [K_i]$, in which $[K_i]$ represents the (non-trivial) knot-cobordism class of the link of a singular branch, $i = 0, \ldots, n$, had any solutions other than $K_1 = K_0, n = 1$. Litherland, using his calculations of the signatures of iterated torus knots [Li], was able to show that there were only such trivial solutions. It follows that, for instance, there is no family $\{\Gamma_s\}$, $|s| < \varepsilon$, of (local) curves in a small ball in $\mathbf{C}^2$ so that Γ_s for $s \neq 0$ has two singular points each with a single branch, while Γ_0 has only one singularity, locally of the form $z = t^2, w = t^5$. Is there another proof of the non-existence of such a deformation? (Multiplicities would allow two cusps.)

Litherland's formulas, of course, give all the various signatures of the links of singularities (though the expression is in closed form only by the use of a counting function involving "greatest integer in ...", which makes them rather a bore to calculate). If one lowers one's sights, and asks only about the classical signature (that corresponding to the root -1 of unity), and then only about its sign, an easy direct proof—again, using the representation of the link as a closed positive braid—shows that the signature of the link of a singularity is positive, [Ru 5].

Finally, some conjectures on less algebraic knot invariants of links of singularities should be mentioned. The *Milnor number* μ of a singularity is the rank of $H_1(F_0; \mathbf{Z})$. Let us look at a single branch, for convenience. Then Milnor conjectured [Mi 2] that $\mu/2$, which is the genus of F_0 and therefore (by a general theorem about fibred links) the genus of the knot ∂F_0, actually is the *slice genus* of ∂F_0. One can make the weaker conjecture that at least $\mu/2$ is the *ribbon genus* of ∂F_0. Milnor also wondered if this integer equalled the *Überschneidungszahl*, or *gordian number*, of ∂F_0; again the conjecture can be weakened, if one introduces the concepts of "slice Überschneidungszahl" and "ribbon Überschneidungszahl", cf. [Ru 2]. The conjectures are true in various cases where direct calculations can be made (e.g., the cusps $z = t^2$, $w = t^3$), but I know of no general results.

§5. Global knot theory in brief—the projective case

A curve $\Gamma \subset \mathbf{CP}^2$ can be given by its resolution $r: R \to \Gamma$ (a complex-analytic map from a compact Riemann surface into $\mathbf{CP}^2$ which is generically one-to-one on R) or by its polynomial $F(z_0, z_1, z_2) \in \mathbf{C}[z_0, z_1, z_2]$ (the homogeneous polynomial of least degree, not identically zero, which vanishes at every point of Γ). These suggest different kinds of knot-theoretical questions. One can consider all curves with diffeomorphic resolutions (the requirement that the curves have complex-analytically equivalent resolutions would be too stringent, and is less topological), and ask how differently they can be placed in the plane. Or one can consider families of curves, each cut out by a polynomial of some fixed degree.

Let P_d denote the projective space of the vector space of homogeneous complex polynomials in (z_0, z_1, z_2) of degree d. Because we never want to consider curves with multiple components, we throw out of P_d the algebraic subset corresponding to reducible polynomials with a multiple factor; the remaining Zariski-open subset Q_d corresponds to the set of what we may call curves of *geometric degree* d. If (the equivalence class of) $F(z_0, z_1, z_2)$ belongs to P_d, let $\Gamma_F = \{(z_0 : z_1 : z_2) \in \mathbf{CP}^2 : F(z_0, z_1, z_2) = 0\}$; then $F \in Q_d$ if and only if there is an open dense set of lines in $\mathbf{CP}^2$ which intersect Γ_F transversely in d distinct points.

The condition that Γ_F have a singular point is, of course, an algebraic condition on F. Let $S_d \subset P_d$ be the algebraic subset of singular curves without multiple components, and $R_d = Q_d - S_d$ the Zariski-open subset of polynomials of *geometrically regular curves* of geometric degree d. Any curve Γ_F, $F \in R_d$, is its own resolution ($r =$ identity). By connecting any two $F, G \in R_d$