

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 29 (1983)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SOME KNOT THEORY OF COMPLEX PLANE CURVES
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DOI: <https://doi.org/10.5169/seals-52979>

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SOME KNOT THEORY OF COMPLEX PLANE CURVES ¹⁾

by Lee RUDOLPH ²⁾

§1. ASPECTS OF THE “PLACEMENT PROBLEM” FOR COMPLEX PLANE CURVES

How can a complex curve be placed in a complex surface?

The question is vague; many different ways to make it more specific may be imagined. The theory of deformations of complex structure, and their associated moduli spaces, is one way. Differential geometry and function theory, curvatures and currents, could be brought in. Even the generalized Nevanlinna theory of value distribution, for analytic curves, can somehow be construed as an aspect of the “placement problem”.

By “knot theory” I mean to connote those aspects of the situation that are more immediately topological. I hope to show that there is something of interest there.

§2. A TRIPTYCH

Here are three ways to interpret the phrase “knot theory of complex plane curves”.

Globally: the “complex plane” is projective space \mathbf{CP}^2 or affine space \mathbf{C}^2 ; a “curve” is an algebraic curve (in projective space) or an algebraic or analytic curve (in affine space); here, “knot theory” has historically been largely concerned with studying the “knot group”, though there are also results on “knot type”.

Locally: a “complex plane curve” is the germ of a plane curve (algebraic, analytic, or formal) over \mathbf{C} ; this is the study of singularities, and “knot theory” has been the classical knot theory of links in the 3-sphere, put to work in the service of that study.

In between: a “complex plane curve” is an analytic curve in a reasonable open set in a complex surface (chiefly, in the theory as so far developed, the

¹⁾ This article has already been published in *Nœuds, tresses et singularités*, Monographie de l’Enseignement Mathématique N° 31, Genève 1983, p. 99-122.

²⁾ Research partially supported by NSF Grant MCS 76-08230.

interior of a ball or a bidisk), well-behaved at the boundary; a knot-theorist can study either of two codimension-2 situations—the complex curve in its ambient space, or the boundary of this pair.

This middle panel of the triptych has been less studied than the other two, though it is of obvious relevance to both.

§3. RÉSUMÉ OF BASIC DEFINITIONS

By *complex surface* I mean a smooth manifold of 4 real dimensions, equipped with a complex structure. A *complex curve* Γ in a complex surface M is a closed subset which is locally of the form $\{(z, w) \in U \subset \mathbf{C}^2 : f(z, w) = 0\}$ where $f : U \rightarrow \mathbf{C}$ is a nonconstant complex analytic function. A *Riemann surface* is a smooth manifold of 2 real dimensions, equipped with a complex structure.

It is a fundamental fact, to which is due the especial appositeness of classical knot theory to the study of curves in surfaces, that any complex curve $\Gamma \subset M$ has a *resolution* of the following sort: There is a Riemann surface R , and a holomorphic mapping $r : R \rightarrow M$, so that $r(R) = \Gamma$; in fact, there is a discrete (possibly empty) subset $\mathcal{S}(\Gamma) \subset \Gamma$, the *singular locus of Γ in M* , so that the *regular locus* $\mathcal{R}(\Gamma) = \Gamma - \mathcal{S}(\Gamma)$ is a Riemann surface, and R is the union (with what turns out to be a unique topology and complex structure) of $\mathcal{R}(\Gamma)$, on which r is the identity, and a discrete set $r^{-1}(\mathcal{S}(\Gamma)) \subset R$ mapping finitely-to-one onto $\mathcal{S}(\Gamma)$.

The singular locus is, of course, exactly the set of points of Γ at which, no matter what the local representation of Γ as the zeroes of an analytic function $f(z, w)$, the (complex) gradient vector ∇f vanishes.

If P is a point of Γ , and $Q \in r^{-1}(P) \subset R$, then the germ at P of the r -image of a small disk on R centered at Q is called a *branch* of Γ at P . (Abusively, “branch” may also be used below to refer to some representatives of this germ.) Naturally, at a regular point there is only one branch; but there may be only one branch at a point, and the point still be singular.

References: [G-R], [Mi 2].

§4. LOCAL KNOT THEORY IN BRIEF

Using local coordinates in the resolution R and the ambient surface M , one sees that each branch of a curve Γ can be parametrized either by $z = t, w = 0$ or (more interestingly) by some pair $z = t^m, w = t^n + c_{n+1}t^{n+1} + \dots$

$+ c_N t^N$, $t \in \mathbf{C}$, with $n > m$. (In the original choice of coordinates, r might well involve genuine power series; but it is not hard to make a formal change of coordinates to one of the forms above, involving only polynomials; and it is not much harder to prove a comparison theorem, the remote ancestor of that of M. Artin, which shows that actually the formal change of coordinates can be taken to be somewhere convergent.) Consider the “approximations” to such a branch, gotten by dropping all terms of w from some degree on up: so the first approximation is (t^m, t^n) , and the $(N-n+1)$ st is the branch we began with. Each of these is itself a map onto a branch of some curve; generally not one-to-one.

Define integers $g(1), \dots, g(N-n+1)$ by saying that the k -th approximation is $g(k)$ -to-one in a punctured neighborhood of $t = 0$. Then $g(1) = \text{GCD}(m, n)$, $g(k+1)$ divides $g(k)$, and $g(N-n+1) = 1$. These integers can be calculated as follows. Let $\mathbf{C}[[t]]$ be the algebra of formal power series, with unique maximal ideal $m = t\mathbf{C}[[t]]$. Let A_k be the m -adically closed subalgebra generated by 1 and the components of the k -th approximation. Then $g(k)$ is the least integer g such that $A_k \subset \mathbf{C}[[t^g]] \subset \mathbf{C}[[t]]$. (One gets the same answer starting from the algebra $\mathbf{C}\{t\}$ of somewhere-convergent power series.) A parametrization of the branch covered by the k -th approximation is $z = t^{m/g(k)}$, $w = t^{n/g(k)} + \dots + c_{n+k-1} t^{(n+k)/g(k)}$.

The knots in which we are interested arise when we intersect the branch under investigation with the boundary of an infinitesimal 4-disk containing the singular point. The 4-disk used may be either a *round disk* $D_\varepsilon^4 = \{(z, w) : |z|^2 + |w|^2 = \varepsilon^2\}$ with boundary the *round sphere* S_ε^3 , or a *bidisk* $D(\varepsilon_1, \varepsilon_2) = \{(z, w) : |z| \leq \varepsilon_1, |w| \leq \varepsilon_2\}$, with boundary comprised of two solid tori $\partial_1 D(\varepsilon_1, \varepsilon_2) = \{|z| = \varepsilon_1, |w| \leq \varepsilon_2\}$ and $\partial_2 D(\varepsilon_1, \varepsilon_2) = \{|z| \leq \varepsilon_1, |w| = \varepsilon_2\}$ which together make up a 3-sphere with corners. Whether one uses round disks or bidisks, one obtains a knot of the same type. The bidisk boundary is more convenient here, when we are studying the branch parametrically; from the assumption that $n > m$ we can see that, for sufficiently small $\varepsilon > 0$, the branch intersects $\partial D(\varepsilon, \varepsilon)$ only along $\partial_1 D(\varepsilon, \varepsilon)$.

The first approximation to the branch actually meets $\partial_1 D$ on the torus $\{|z| = \varepsilon, |w| = \varepsilon^{n/m}\}$, where it covers, $g(1)$ to one, a *torus knot* of type $O\{m/g(1), n/g(1)\}$. (Here is the notation I am using, cf. [Ru 4]. If K is any oriented knot in an oriented 3-sphere, with closed tubular neighborhood $N(K)$, let L be an oriented simple closed curve on $\partial N(K)$ which is not null-homologous on this torus; then there are relatively prime integers p and q so that L has linking number q with K and represents p times the class of K in $H_1(N(K); \mathbf{Z})$. We then call L a *cable of type (p, q) about K* and denote it by $K\{p, q\}$. When cabling is iterated, excess curly braces become semicolons. The unknot is denoted by O ; a

cable about the unknot is also called a *torus knot*; a cable about ... a cable about the unknot is an *iterated torus knot*.) This knot type does not change when ε is made smaller.

Now suppose that for all sufficiently small $\varepsilon > 0$, the k -th approximation to a branch intersects $\partial D(\varepsilon, \varepsilon)$ in a knot of type $O\{p_1, q_1; \dots; p_k, q_k\}$. Considering how we pass to the next approximation we see that there are relatively prime integers p_{k+1} and q_{k+1} so that, for all sufficiently small $\varepsilon > 0$, the $(k+1)$ -st approximation to the branch intersects $\partial D(\varepsilon, \varepsilon)$ in a knot of type $O\{p_1, q_1; \dots; p_k, q_k; p_{k+1}, q_{k+1}\}$. (The difference between successive approximations is 0 or a monomial $c_{n+k}t^{n+k} \not\equiv 0$, which contributes an “epicycle” that for small enough ε precisely creates a cabling.) In fact, $p_{k+1} = g(k)/g(k+1)$ (note that for any K and q , $K\{1, q\}$ is the same knot type as K); the formula for q_{k+1} is more complicated, and we won’t give it.

Consider a curve with a singular point at which there are two or more branches. Coordinates in the ambient surface can be chosen so that each branch differs only by a diagonal linear transformation in (z, w) from one of the form just studied (including the non-singular case $z = t, w = 0$). Each branch individually contributes an iterated torus knot to the *link of the singularity*, $\Gamma \cap \partial D(\varepsilon, \varepsilon)$; and in fact they all fit together nicely. An elegant description of how they do is given in [E-N]; see also, and for this section generally, [Lê] and [Mi 2] and references cited therein.

After Burau, Zariski, *et al.*, had established that any point of a curve in a (non-singular) surface had local topology that was completely described by this link-type invariant, the strictly topological investigation of singular points seems to have languished for some decades. (The algebraic geometers, of course, had also established that this link-type invariant—more precisely, the sequences of pairs (p_i, q_i) for each branch, and the linking numbers between the iterated torus knots of different branches, from which the whole link of the singularity can be reconstructed—was equivalent to some numerical invariants which had long been known and which could be detected purely algebraically, namely, the Puiseux pairs of the various branches and the intersection multiplicity of the pairs of branches. They also pressed forward with their investigations of continuous invariants within the family of singularities of a given link type. But that is another story.) However, in the late 1960’s, Milnor [Mi 2] gave new life to the subject when he showed that the link of a singularity was a “fibred”, or Neuwirth-Stallings, link.

Milnor’s proof uses the round-sphere model. He shows that, if $\Gamma \subset \mathbf{C}^2$ is the zero-locus of $p(z, w) \in \mathbf{C}[z, w]$, $p(0, 0) = 0$, then for all sufficiently small $\varepsilon > 0$, the restriction ϕ of the map $\arg p : \mathbf{C}^2 - \Gamma \rightarrow S^1 : (z, w) \mapsto p(z, w)/|p(z, w)|$ to

$S_\varepsilon^3 - \Gamma$ is the projection map of a fibration over S^1 . The fibre is diffeomorphic to the interior of the surface $F_0 = S_\varepsilon^3 \cap \{(z, w) : p(z, w) \text{ is real and non-negative}\}$. (Note that the change in viewpoint from bidisk boundary to round sphere is accompanied by a change from branch-as-parametrized-disk to branch-as-level-set.)

We will see below that the link of a singularity is in a natural way a closed strictly positive braid; I will give a geometric proof of the well-known fact that such a closed braid is a fibred link.

Inspired by Milnor's Fibration Theorem, a number of mathematicians began investigations of knot-theoretical properties of the links of singularities. The fibration ϕ determines an autodiffeomorphism of F_0 (fixed on the boundary), unique up to isotopy relative to the boundary, which is variously called the *characteristic map*, *holonomy*, or *monodromy* of the fibration; it induces an automorphism (also called the monodromy) of the integral homology of F_0 . From the homology monodromy one can calculate the Alexander polynomial of the link of the singularity; this was done in [Lê], where it was also shown that two branches defined iterated torus knots in the same knot-cobordism class if and only if they defined knots of the same knot type, the proof following from a study of the roots of the Alexander polynomials.

I wondered how independent these distinct knot-cobordism classes might be, in the knot-cobordism group; in particular, I asked [Ru 6] whether the equation $[K_0] = \sum_{i=1}^n [K_i]$, in which $[K_i]$ represents the (non-trivial) knot-cobordism class of the link of a singular branch, $i = 0, \dots, n$, had any solutions other than $K_1 = K_0, n = 1$. Litherland, using his calculations of the signatures of iterated torus knots [Li], was able to show that there were only such trivial solutions. It follows that, for instance, there is no family $\{\Gamma_s\}, |s| < \varepsilon$, of (local) curves in a small ball in \mathbf{C}^2 so that Γ_s for $s \neq 0$ has two singular points each with a single branch, while Γ_0 has only one singularity, locally of the form $z = t^2, w = t^5$. Is there another proof of the non-existence of such a deformation? (Multiplicities would allow two cusps.)

Litherland's formulas, of course, give all the various signatures of the links of singularities (though the expression is in closed form only by the use of a counting function involving "greatest integer in ...", which makes them rather a bore to calculate). If one lowers one's sights, and asks only about the classical signature (that corresponding to the root -1 of unity), and then only about its sign, an easy direct proof—again, using the representation of the link as a closed positive braid—shows that the signature of the link of a singularity is positive, [Ru 5].

Finally, some conjectures on less algebraic knot invariants of links of singularities should be mentioned. The *Milnor number* μ of a singularity is the rank of $H_1(F_0; \mathbf{Z})$. Let us look at a single branch, for convenience. Then Milnor conjectured [Mi 2] that $\mu/2$, which is the genus of F_0 and therefore (by a general theorem about fibred links) the genus of the knot ∂F_0 , actually is the *slice genus* of ∂F_0 . One can make the weaker conjecture that at least $\mu/2$ is the *ribbon genus* of ∂F_0 . Milnor also wondered if this integer equalled the *Überschneidungszahl*, or *gordian number*, of ∂F_0 ; again the conjecture can be weakened, if one introduces the concepts of “slice Überschneidungszahl” and “ribbon Überschneidungszahl”, cf. [Ru 2]. The conjectures are true in various cases where direct calculations can be made (e.g., the cusps $z = t^2$, $w = t^3$), but I know of no general results.

§5. GLOBAL KNOT THEORY IN BRIEF—THE PROJECTIVE CASE

A curve $\Gamma \subset \mathbf{CP}^2$ can be given by its resolution $r: R \rightarrow \Gamma$ (a complex-analytic map from a compact Riemann surface into \mathbf{CP}^2 which is generically one-to-one on R) or by its polynomial $F(z_0, z_1, z_2) \in \mathbf{C}[z_0, z_1, z_2]$ (the homogeneous polynomial of least degree, not identically zero, which vanishes at every point of Γ). These suggest different kinds of knot-theoretical questions. One can consider all curves with diffeomorphic resolutions (the requirement that the curves have complex-analytically equivalent resolutions would be too stringent, and is less topological), and ask how differently they can be placed in the plane. Or one can consider families of curves, each cut out by a polynomial of some fixed degree.

Let P_d denote the projective space of the vector space of homogeneous complex polynomials in (z_0, z_1, z_2) of degree d . Because we never want to consider curves with multiple components, we throw out of P_d the algebraic subset corresponding to reducible polynomials with a multiple factor; the remaining Zariski-open subset Q_d corresponds to the set of what we may call curves of *geometric degree* d . If (the equivalence class of) $F(z_0, z_1, z_2)$ belongs to P_d , let $\Gamma_F = \{(z_0 : z_1 : z_2) \in \mathbf{CP}^2 : F(z_0, z_1, z_2) = 0\}$; then $F \in Q_d$ if and only if there is an open dense set of lines in \mathbf{CP}^2 which intersect Γ_F transversely in d distinct points.

The condition that Γ_F have a singular point is, of course, an algebraic condition on F . Let $S_d \subset P_d$ be the algebraic subset of singular curves without multiple components, and $R_d = Q_d - S_d$ the Zariski-open subset of polynomials of *geometrically regular curves* of geometric degree d . Any curve Γ_F , $F \in R_d$, is its own resolution ($r = \text{identity}$). By connecting any two $F, G \in R_d$

with a path in R_d , one may construct an isotopy (which may be effected by an ambient isotopy) between the curves Γ_F and Γ_G in \mathbf{CP}^2 ; so all these curves are diffeomorphic, and of the same knot type in the plane. More generally, $F \in Q_d$ lies in a maximal connected subset of Q_d of polynomials G such that Γ_F and Γ_G are ambient isotopic, through algebraic curves. These subsets form a stratification of Q_d which is little understood. Zariski [Z] showed that two (singular) curves in Q_6 , homeomorphic and with the same type and number of singularities (cusps), were not in the same stratum, by showing that the knot groups $\pi_1(\mathbf{CP}^2 - \Gamma_F)$ and $\pi_1(\mathbf{CP}^2 - \Gamma_G)$ were not isomorphic. In general, as we will see below, the knot group cannot distinguish strata.

An interesting question (I do not know to whom it is due: I heard of it in Dennis Sullivan's problem seminar at M.I.T. in the summer of 1974) is whether there are curves Γ_F and Γ_G which are ambient isotopic but *not* so through algebraic curves. I know of no results here.

The incidence structure of this stratification of Q_d by "algebraic ambient isotopy types" is, especially, not understood: this is the theory of degenerations. It can be proved that the knot group associated to a given stratum is the homomorphic image of the knot group associated to any stratum incident to the given stratum. Partly, it was the desire to apply this fact to the proof of the Zariski Conjecture (see below) which led investigators for many years to the study of some particular (unions of) strata to which we now turn.

First we recall the two simplest sorts of singularities. A *cusp* has a single branch, locally given by $z = t^2, w = t^3$; the link of a cusp is a trefoil knot (of a fixed handedness once one establishes conventions). A *node* has two branches, each itself nonsingular, with distinct tangent lines; it can be locally given by the equation $zw = 0$, and its link is a Hopf link of two components (linking number +1). A curve Γ is a *node curve* if all its singularities (if any) are nodes, and a *cusp curve* if all its singularities are either nodes or cusps.

We also recall, what we have not needed before, the notion of reducibility: a curve Γ is *reducible* if its resolution is not connected; alternatively Γ_F is reducible if and only if F is reducible but square-free. A curve that is not reducible is *irreducible*.

The extreme of reducibility is displayed by any $F \in Q_d$ which is the product of d linear factors. Then the curve Γ_F is the union of d projective lines, which we will say (here) are in *general position* if Γ_F is a node curve, that is, if no three of the lines are concurrent. Let $L_d \subset Q_d$ be the set of all such completely reducible curves. Then L_d is a single stratum. Let $N_d \subset Q_d$ be the set of polynomials of node curves; N_d is a union of strata. What is now called the Severi Conjecture is the statement that L_d is incident to every stratum in N_d ; in other words, that every

node curve can be degenerated to d lines in general position. We will compute the knot group of d lines in general position below. It is, in particular, abelian. Consequently, the truth of the Severi Conjecture would imply that *the knot group of any node curve is abelian*—a statement long known as the Zariski Conjecture, which has recently been proved true by quite other means [F-H, De]. Of course, independent of the truth of the Severi Conjecture, one can study the union $M_d \subset N_d$ of those strata which actually are incident to L_d . Moishezon [Mo] calls M_d the *mainstream* of node curves in his investigation of “normal forms for braid monodromies”. Such normal forms (when they exist) enrich the datum of the knot group by giving it in a particularly nice presentation related to the algebraic geometry.

Now let $K_d \subset Q_d$ correspond to the cusp curves. Here the knot groups need no longer be abelian. In fact, for

$$F(z_0, z_1, z_2) = z_1^2 z_2^2 + 4z_0(z_2^3 - z_1^3) + 6z_0^2 z_1 z_2 - 27z_0^4$$

in K_4 , a curve with three cusps and no nodes (which has resolution

$$r : \mathbf{CP}^1 \rightarrow \Gamma_F : (t_0 : t_1) \mapsto (t_0^2 t_1^2 : t_1^4 + 2t_0^3 t_1 : 2t_0 t_1^3 + t_0^4),$$

the knot group can be computed (as by Zariski [Z] or, algebraically, by Abhyankar [Ab]) to have the presentation $(a, b : aba = bab, a^4 = 1, a^2 = b^2)$, making it non-abelian of order 12.

The knot groups of cusp curves have been studied because of their application to the study and possible classification of complex (algebraic) surfaces. In fact, if $f : Y \rightarrow \mathbf{CP}^2$ is a so-called stable finite morphism, $\Sigma' \subset Y$ the locus where f is not étale, $\Sigma = f(\Sigma')$, then Σ is a cusp curve.

Zariski commissioned van Kampen, in the early 1930's, to calculate the knot group of an arbitrary curve [vK]; van Kampen gave his solution in terms of a certain presentation of the knot group. If Γ has (geometric) degree d , then van Kampen's presentation has d generators x_1, \dots, x_d which represent loops in a fixed projective line \mathbf{CP}_∞^1 transverse to Γ ; the intersection $\Gamma \cap \mathbf{CP}_\infty^1$ contains d points P_1, \dots, P_d , and x_i is a loop from a basepoint $* \in \mathbf{CP}_\infty^1$ out to P_i , around it once counterclockwise, and back to $*$. One relation is then that $x_1 \dots x_d = 1$. The rest arise by carrying \mathbf{CP}_∞^1 around certain loops of lines. In fact, let \mathbf{CP}^{2*} be the dual projective plane, each point of which is a line in \mathbf{CP}^2 ; and let Γ^* contain all lines which are either tangent to Γ or pass through one of its singular points. Then Γ^* is a curve in \mathbf{CP}^{2*} . If $*$ and \mathbf{CP}_∞^1 are sufficiently general, then the pencil of lines in \mathbf{CP}^2 through $*$, which is itself a line in \mathbf{CP}^{2*} , will be transverse to Γ^* . The (free) fundamental group of the complement of Γ^* in this pencil is naturally represented in the automorphism group of the free group $(x_1, \dots, x_d : x_1 \dots x_d)$.

$= 1$). The rest of the relations needed for the van Kampen presentation of $\pi_1(\mathbf{CP}^2 - \Gamma; *)$ come, then, by declaring this representation trivial. One obtains a finite presentation, of course, by choosing generators of the acting free group; Moishezon's problem of "normal forms" is essentially the problem of making a good choice. Several modernizations [Abe], [Che], [Cha] of van Kampen's proof have been published in recent years.

In a *standard* van Kampen presentation (where the generators of the acting free group are free generators), each relation corresponds either to a singularity of Γ or to a simple vertical tangent to Γ ; and (up to the action of the corresponding free generator) each relation is of a certain canonical form, which depends only on the *closed braid type* (§7) of the link of the branch(es) at the point of Γ , through which the line in the pencil passes that gives the relation in question, where this line itself is used to find the axis of the closed braid. In particular, the knot group of a node curve always has a standard van Kampen presentation in which each relation either sets conjugates of two x_i equal (from a simple vertical tangent) or says that two such conjugates commute (from a node); if "conjugates" could be deleted, the Zariski Conjecture would be trivially true.

There is also a great body of work on "knot groups" of curves in (compact, smooth) complex surfaces other than \mathbf{CP}^2 , and on the related issue of fundamental groups of surfaces; we cannot touch on these topics here.

§6. GLOBAL KNOT THEORY IN BRIEF—THE AFFINE CASE

Little appears to be known about algebraic curves in affine space, from the knot-theoretical viewpoint. The gross algebraic topology (even just homology theory) of \mathbf{CP}^2 is implicated with the quite rigid geometry; but affine space is contractible, and on the other hand its geometry is "infinite" (for instance in the sense that there are Lie groups of arbitrarily high dimension contained in the group of biregular automorphisms of \mathbf{C}^2), so that the conspirators have fallen out and neither can give away much about the other.

One might think, for example, to study the embedding of a curve Γ in \mathbf{C}^2 by first embedding \mathbf{C}^2 itself into \mathbf{CP}^2 . Then the affine complement $\mathbf{C}^2 - \Gamma$ becomes the projective complement $\mathbf{CP}^2 - (\Gamma \cup \mathbf{CP}_\infty^1)$, where $\Gamma \cup \mathbf{CP}_\infty^1$ is a (reducible) projective algebraic curve. The obstacle to this program is the unfortunate fact that \mathbf{C}^2 , just as an algebraic surface, without distinguished coordinates, is not uniquely embedded as $\mathbf{CP}^2 - \mathbf{CP}_\infty^1$. Any biregular automorphism of \mathbf{C}^2 (in particular, one of the vast majority which cannot be extended biregularly to \mathbf{CP}^2)

will move Γ around, and so the configuration of $\Gamma \cup \mathbf{CP}_\infty^1$ is not determined by the embedding of Γ in \mathbf{C}^2 . (For instance, though the geometric number of points at infinity on Γ is determined by Γ , the algebraic intersection number of the closure of Γ with the line at infinity can be made arbitrarily large. Likewise the local singularities at infinity are not determined by the affine curve.)

The main theorems known here have been proved by Abhyankar and his collaborators [A-M, A-S]. They are unknotting theorems, in the sense that they take this form: "Let Γ be a certain curve in \mathbf{C}^2 , and let $i: \Gamma \rightarrow \mathbf{C}^2$ be any algebraic embedding; then there is a biregular automorphism of \mathbf{C}^2 returning i to the inclusion map". Briefly, such a curve Γ cannot be knotted in \mathbf{C}^2 .

However, for most of the curves they deal with, these theorems are not genuinely topological, for the reembedding i is required to be an embedding of Γ with its given structure as a variety, and generally there might be moduli. Only in the original theorem [A-M] (which had been stated, but not correctly proved, by Segre) are there no conceivable moduli, when Γ is a straight line. Then the theorem is this.

THEOREM. *Let $\Gamma \subset \mathbf{C}^2$ be an algebraic curve without singularities, homeomorphic to \mathbf{C} . Then there is a biregular change of coordinates $A: \mathbf{C}^2 \rightarrow \mathbf{C}^2$ so that $A\Gamma$ is a straight (complex) line.*

A topological proof has been given in [Ru 4]. It goes like this. One shows (just as for a singular point) that the intersection of Γ (which we can assume to be parametrized by $z = p(t)$, $w = q(t)$, $p, q \in \mathbf{C}[t]$) with a *very large* bidisk boundary is an iterated torus knot $K = O\{m_1, n_1; \dots; m_s, n_s\}$, with $m_1 = m/GCD(m, n)$, $n_1 = n/GCD(m, n)$, $m = \deg p$, $n = \deg q$. By hypothesis, K is a slice knot. This forces $K = O$, in particular, one of m_1, n_1 is 1. Thereafter the argument is as in [A-M]—if (say) $m_1 = 1$ and p and q are monic, then the biregular change of coordinates $(z, w) \mapsto (z, w - z^{m/n})$ carries Γ to another curve satisfying the hypotheses, of lower bidegree; and so we proceed until one of z, w is linear and the other constant.

As to analytic curves in affine space, almost nothing is known. The obvious analogue of the Theorem above is definitely false: for it is known that the unit disk in \mathbf{C} can be properly analytically embedded in \mathbf{C}^2 [H]; since the disk and the line are analytically inequivalent, no analytic change of coordinates in \mathbf{C}^2 could unknot the disk to a line. It is, however, perfectly possible that every such disk is smoothly unknotted. Presently I am unable even to prove that an analytic line in \mathbf{C}^2 is smoothly unknotted.

§7. THE MIDDLE RANGE

We return, as at the beginning of §4, to the study of intersections of curves in \mathbf{C}^2 with round disks D_r^4 and their boundaries S_r^3 , and bidisks $D(r_1, r_2)$ and their boundaries. Now the (bi)radii are no longer required to be very small.

An embedding $i : (S, \partial S) \rightarrow (D_r^4, S_r^3)$ of a surface-with-boundary S into a round disk is a *ribbon embedding* provided that $N \circ i$ is a Morse function without local maxima on $\text{Int } S$, where $N(z, w) = |z|^2 + |w|^2$; and a surface-with-boundary $S \subset D_r^4$, with $\partial S = S_r^3 \cap S$, is a *ribbon surface* if the inclusion $(S, \partial S) \subset (D_r^4, S_r^3)$ is isotopic through embeddings of pairs to a ribbon embedding. To demand that a surface be ribbon is to place genuine topological restrictions on the embedding.

A theorem of Milnor [Mi 1], specialized to our dimensions, shows that if $\Gamma \subset \mathbf{C}^2$ is a nonsingular analytic curve then for almost all choices of origin and radius, the inclusion of $(\Gamma \cap D_r^4, \Gamma \cap S_r^3)$ into (D_r^4, S_r^3) is a ribbon embedding. A continuity argument easily shows that for no matter what choice of origin, $N|_{\Gamma}$ has critical points, possibly degenerate, of index no greater than 1. It is easy to see that if Γ has singularities, an analogous theorem holds for $N \circ r : R \rightarrow [0, \infty[$ on the resolution. All these results generalize the Maximum Modulus Principle. Nothing much more seems to be known about big round disks and complex plane curves.

Turning our attention to bidisks, we let the way that they separate the variables z and w suggest an attitude to adopt towards our curves: consider one variable (conventionally w) as an analytic but possibly multiple-valued function of the other.

More precisely, let E_n be the space of unordered n -tuples of points of \mathbf{C} (duplications allowed). Then E_n inherits a topology, and a structure of algebraic variety (affine, and singular if $n \geq 2$), from its description as $\mathbf{C}^n / \mathcal{S}_n$, where the symmetric group \mathcal{S}_n acts by permuting coordinates. Let E_n keep its topology, but normalize and resolve its algebraic variety structure, by using the map $\mathbf{C}^n \rightarrow E_n$ which carries (c_1, \dots, c_n) to $\{r_1, \dots, r_n\}$ such that $(w - r_1) \dots (w - r_n) = w^n + c_1 w^{n-1} + \dots + c_n$. Now any function $F : X \rightarrow E_n$ can be called an *n-valued (complex) function on X* . The *graph* of an n -valued function on X is the obvious subset of $X \times \mathbf{C}$; adjectives like “continuous”, “analytic”, “algebraic” apply to n -valued functions in the obvious way.

We make the convention that (if X is not discrete) the entire image $F(X)$ should not lie in the subset $\Delta \subset E_n$ of unordered n -tuples with at least one duplication; Δ is an algebraic hypersurface (irreducible, and singular if $n \geq 3$) in

the affine space E_n , called the *discriminant locus*. Its complement $E_n - \Delta$ is called the *configuration space* (of n distinct points in \mathbf{C}).

To allow infinity as a value, we could replace \mathbf{C} by \mathbf{CP}^1 , E_n by \mathbf{CP}^n , and so on.

Let $f(z, w) \equiv f_0(z)w^n + f_1(z)w^{n-1} + \dots + f_n(z) \in \mathbf{C}[z, w]$. Historically [Bl], the equation $f(z, w) = 0$ (or equivalently the curve it defines) was said to give w as an *algebraic function* of z , provided only that $f(z, w)$ was without repeated factors and without factors of the form $z - c$. (Also, of course, $f_0(z) \not\equiv 0$.) Then, in fact, on the complement in \mathbf{C} of the zero-locus of $f_0(z)$, the assignment $z \mapsto \{w : f(z, w) = 0\}$ is an algebraic n -valued complex function. A zero of $f_0(z)$ is called a *pole* of the algebraic function, and can be accounted for by letting infinity be a value.

If $f_0(z) \equiv 1$, so that there are no poles at all, the algebraic function is *entire*. More generally, if $f_0(z), \dots, f_n(z)$ are allowed to be entire functions of z (in the usual sense), then $f(z, w) = 0$ gives w as an n -valued *meromorphic function*; and if also $f_0(z) \equiv 1$, w is an *entire analytic n-valued function*. The graph of an n -valued entire function is a curve (algebraic or analytic as the case may be); when there are poles, the graph must be closed up to provide fibres over them.

Conversely, any algebraic curve in \mathbf{C}^2 becomes, after almost any linear change of coordinates, such a graph for some n . (This is not so for analytic curves, in general.) Thus we can study plane curves by studying certain curves in E_n .

Let $\gamma \subset \mathbf{C}$ be a simple closed curve, R the compact simply-connected region it bounds, $F : R \rightarrow E_n$ a continuous n -valued function analytic on $\text{Int } R$ with $F(\gamma) \cap \Delta = \emptyset$. Then there is some radius $M > 0$ so that the graph of $F|_\gamma$ lies in the open solid torus $\gamma \times \{w \in \mathbf{C} : |w| < M\}$; and this graph is a (not necessarily connected) n -sheeted covering space of γ . An application of one version of the Maximum Modulus Principle [G-R] shows that actually the graph of F itself is contained in $D = R \times \{w : |w| \leq M\} \subset \mathbf{C}^2$, a topological 4-ball (with boundary 3-sphere piecewise as smooth as γ). Now, $F(R) \cap \Delta$ must be finite; let $F^{-1}(\Delta) \subset R$ be called the *branch locus*, and denoted B . One can easily see that the graph of F in D is a 2-dimensional pseudomanifold-with-boundary (i.e., geometric relative cycle), with any singularities lying in $B \times \{w : |w| < M\} \subset \text{Int } D$; its boundary in ∂D is exactly the link L which is the graph of $F|_\gamma$. Furthermore, the graph of F is naturally oriented (by its complex structure at the regular points), so L has a natural orientation, and the projection $L \rightarrow \gamma$ preserves orientations.

At this point it is convenient to introduce braids; a general reference is [Bi]. The *braid group on n strings* is the fundamental group $B_n = \pi_1(E_n - \Delta; *)$ of the configuration space. Let $l : [0, 2\pi] \rightarrow E_n - \Delta$, $l(0) = l(2\pi)$, be a parametrization of a loop in the configuration space. Then the graph of l in

$[0, 2\pi] \times \mathbf{C}$ is a *geometric braid*, that is, the union of disjoint arcs, on which pr_1 is a covering projection to $[0, 2\pi]$, and such that the unordered n -tuples of top and bottom endpoints are identical; each arc is called a *string*. Under the map $[0, 2\pi] \times \mathbf{C} \rightarrow S^1 \times \mathbf{C} : (\theta, w) \mapsto (e^{i\theta}, w)$, a geometric braid is carried to a *closed braid* in the open solid torus. When $S^1 \times \mathbf{C}$ is identified with the tubular neighborhood of an unknotted circle in S^3 , in such a way that distinct circles $S^1 \times \{z_0\}$ and $S^1 \times \{z_1\}$ are (algebraically, and therefore geometrically) unlinked, then any closed braid becomes a knot or link in S^3 , and it is naturally oriented. For $\beta \in B_n$, any closed braid constructed in this way from a loop which represents β is denoted $\hat{\beta}$. If, conversely, $L \subset S^1 \times \mathbf{C}$ is an oriented link on which pr_1 is an orientation-preserving n -sheeted covering map, then any choice of a basepoint $e^{i\theta} \in S^1$ yields a loop in $E_n - \Delta$, based at $* = \{w \in \mathbf{C} : (e^{i\theta}, w) \in L\}$, and thus a braid $\check{L} \in B_n = \pi_1(E_n - \Delta; *)$, with $(\check{L})^\wedge = L$.

Since Δ is irreducible, the abelianization of B_n is infinite cyclic, and in fact B_n is normally generated by one element, that is, generated by a single conjugacy class. Choose for the basepoint $*$ of $E_n - \Delta$ the (real) n -tuple $\{1, \dots, n\}$. Let

$$g_i(z, w) \equiv \left(w^2 - (2i+1)w + \left(i^2 + i + \frac{1}{4}(1-z) \right) \right) \cdot \prod_{\substack{j=1 \\ j \neq i, i+1}}^{n-1} (w-j) \in \mathbf{C}[z, w],$$

for $i = 1, \dots, n-1$; and let $G_i : \mathbf{C} \rightarrow E_n$ be the n -valued function corresponding to $g_i(z, w) = 0$. If $R = \{z : |z| \leq 1\}$, then each $G_i|_R$ is an embedding of R as a normal disk to Δ (at a regular point), with center

$$G_i(0) = \left\{ 1, \dots, i-1, i + \frac{1}{2}, i + \frac{1}{2}, i+2, \dots, n \right\}$$

on Δ , and basepoint $G_i(1) = *$. Giving ∂R its positive (counterclockwise) orientation, we get oriented loops in $E_n - \Delta$, and the homotopy class of $G_i(\partial R)$ is denoted by σ_i and called the *i-th standard generator of B_n* . (The geometric braids corresponding to the given construction are the standard pictures of the σ_i .) The set of standard generators does, in fact, generate B_n , cf. [Bi]. Each σ_i is conjugate to σ_i . Following [Ru 2], let any braid in B_n conjugate to σ_1 be called a *positive band in B_n* ; a loop in the configuration space represents a positive band if and only if it is the oriented boundary of an oriented disk in E_n which meets the discriminant locus transversely in a single positive (regular) point. The inverse of a positive band is a *negative band*.

An ordered k -tuple $\mathbf{b} = (b(1), \dots, b(k))$ of bands in B_n is a *band representation of length k of the braid $\beta(\mathbf{b}) = b(1) \cdots b(k)$* . (A braid word is a band representation where each band is a standard generator or the inverse of a

standard generator.) Each braid has many band representations, corresponding to the various null-homotopies, transverse to Δ , of a loop representing the braid in $E_n - \Delta$ to a point in E_n . (See [Ru 2] for a precise statement and proof.) Such a null-homotopy gives a map of a disk into E_n , transverse to Δ —the length of any corresponding band representation is the geometric number of intersections of the disk with Δ , and the number of positive (resp., negative) bands is the number of positive (resp., negative) intersections with Δ . In particular, suppose each such intersection is positive, so each band $b(s)$ is positive. Then \mathbf{b} , $\beta(\mathbf{b})$, and the closed braid $\hat{\beta}(\mathbf{b})$ are all called (in [Ru 1-4]) *quasipositive*. The closed braid L , associated to an analytic n -valued function F and a simple closed curve γ which bounds a simply-connected region in the domain of F , is quasipositive. (If F as given is not transverse to Δ in R , almost any small translation of F in E_n will become so, while the braid type of L won't change; and complex analytic intersections are positive.)

Conversely, it is shown in [Ru 1] that for every quasipositive band representation \mathbf{b} in B_n , there are an algebraic n -valued function and simple closed curve yielding the given band representation in the manner just exposed. It is also shown (and this is why we have excluded poles) that any type of closed braid whatever can occur as the graph over S^1 of a meromorphic (algebraic) n -valued function on \mathbf{C} . (But note that when poles actually do occur inside the simple closed curve, the closed braid is never the complete boundary of the piece of analytic curve inside a bidisk; a typical example is given by $f(z, w) \equiv zw - \frac{1}{4}$, in $D(1, 1)$, $\gamma = S^1$.)

Let $e : B_n \rightarrow \mathbf{Z}$ be abelianization. Thus $e(\beta)$ is the *exponent sum* of β , when β is written as a braid word in the standard generators; or more generally it is the number of positive bands in \mathbf{b} , minus the number of negative bands in \mathbf{b} , when $\beta(\mathbf{b}) = \beta$. Geometrically, $e(\beta)$ is the linking number of (any loop representing) β with Δ , in E_n . Analytically, $e(\beta)$ can be given by an integral formula, as by Laufer [Lau], where it is called *self-winding* (and is generalized to links that aren't necessarily given as closed braids).

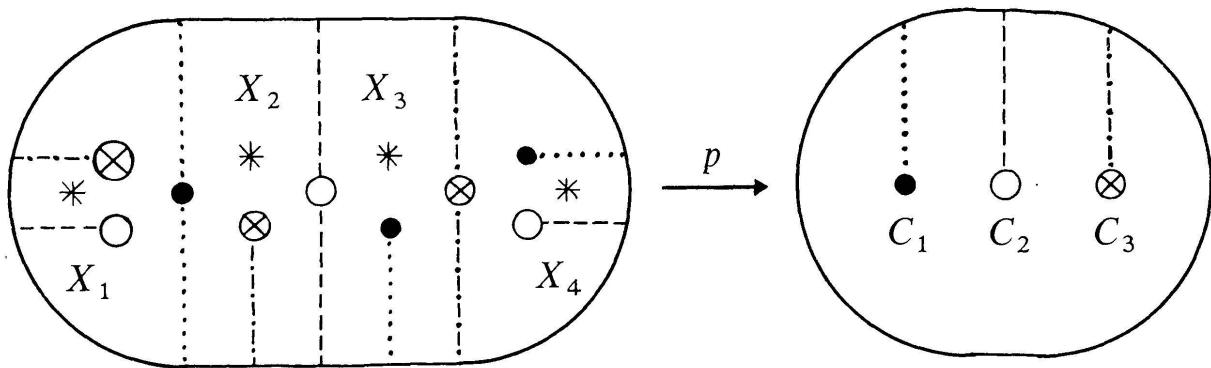
When \mathbf{b} is quasipositive, $e(\beta(\mathbf{b}))$ is the length of \mathbf{b} , a fact with the following geometric meaning. When $F : R \rightarrow E_n$ is smooth and transverse to Δ , then the graph of F is a smooth surface in $R \times \mathbf{C}$; the intersections with Δ correspond to “simple vertical tangents” to the graph, and projection from the graph of F back to R is a branched covering, with only two sheets coming together over each branch point in R . Thus the Euler characteristic $\chi(\text{graph } F)$ equals $n\chi(R) - l$, if l is the number of branch points. When R is a disk and F corresponds to a quasipositive band representation \mathbf{b} , then l is the length of \mathbf{b} and we recover a

genus formula for the graph of F in terms of n , the number of components of the boundary of the graph, and the exponent sum of the boundary. More generally, when F is analytic, even if it is not transverse to Δ it will have a well-defined positive intersection multiplicity at each point of intersection, which will equal the number of geometric intersections of almost any small (analytic) perturbation of F ; thus its graph, which will now be a singular curve, will have well-defined multiplicities for each singular point, and again a genus formula can be recovered, this time involving also these multiplicities: cf. [Lau].

A very interesting subclass of the quasipositive braids consists of the *positive braids*. A braid in B_n is positive if it can be written as a word in the standard generators without using their inverses, *strictly positive* if each of $\sigma_1, \dots, \sigma_{n-1}$ actually occurs. Positive braids play an important algebraic role in the braid group (cf. [Bi]). Closed positive braids enjoy various nice knot-theoretical properties (cf. [St], [Ru 5]), and have turned up in diverse contexts—as knotted orbits of some special dynamical systems [Bi-W]; and, what is relevant here, as the links of singular points of plane curves.

Let $f(z, w) \in \mathbf{C}[z, w]$ be squarefree, not divisible by z , and satisfy $f(0, 0) = 0$. Then for $\varepsilon > 0$ sufficiently small, $f(z, w) = 0$ defines an n -valued analytic function $F : \{z : |z| \leq \varepsilon\} \rightarrow E_n$ with $F^{-1}(\Delta) = \{0\}$. Let $w_1(z), \dots, w_n(z)$ be the n numbers in $F(z)$; then it is readily seen that the assignment $z \mapsto \{w_i(z) - w_j(z) : 1 \leq i, j \leq n, i \neq j\}$ is an $n(n-1)$ -valued analytic function. Without loss of generality, we may take n and ε so that $w_1(0) = \dots = w_n(0) = 0$, and $w_i(z) - w_j(z) \neq 0$ for $z \neq 0, |z| \leq \varepsilon$. Now a straightforward calculation shows that for $z \neq 0, |z| \leq \varepsilon$, we have $d(\arg(w_i - w_j))/d(\arg z) > 0$. Consider the closed braid L which is the graph of $F \mid \{z : |z| = \varepsilon\}$, and the link of the singularity of $\{f = 0\}$ at $(0, 0)$. A braid diagram for L may be obtained by projecting its ambient solid torus $S^1 \times \mathbf{C}$ onto $S^1 \times e^{i\theta} \mathbf{R}$ orthogonally; for almost all θ this will be a braid diagram in general position, from which a braid word may be read off in the usual way; and the signs of the crossings are precisely determined as the signs at the appropriate points of $d(\arg(w_i - w_j))/d\theta$. Since $\theta = \arg z$, the link of a singularity is a positive closed braid. In fact, it can be seen to be strictly positive; for if it were not, it would be a split link, in particular it would have components with zero algebraic linking—but the linking number of two components of the link of a singularity is the intersection number of the corresponding branches, and is strictly positive.

It is known that a strictly positive closed braid is a fibred link, cf. [St], [Bi-W], which provides another proof (in this dimension) of Milnor's Fibration Theorem (that the link of a singularity is fibred—Milnor, of course, gives an actual analytic formula for the fibration). Here is a simple proof which

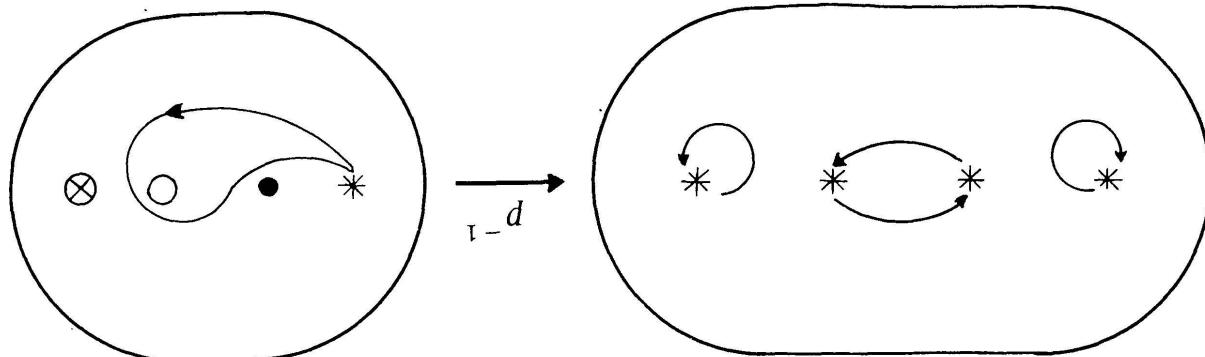
FIGURE 1 ($n=4$)

geometrically constructs a fibration of the complement of a strictly positive closed braid. Let $p : X \rightarrow \mathbf{C}$ be the n -sheeted branched covering with branch locus $\{1, \dots, n-1\}$, where the permutation at j is the transposition $(j \ j+1)$. Then X is homeomorphic to \mathbf{C} again. For concreteness, we realize p as in Figure 1: cuts $C_j = \{w : \operatorname{Re} w = j, \operatorname{Im} w \geq 0\}$ are made in the base space; we coordinate X so that the singular point of $p^{-1}(j)$ is j , and so that $\{z : \operatorname{Re} z = j\}$ is one component of $p^{-1}(C_j)$; then the components of $p^{-1}(\mathbf{C} - \bigcup_{j=1}^{n-1} C_j)$ are the sets

$$X_1 = \{z : \operatorname{Re} z < 1\}, X_2 = \{z : 1 < \operatorname{Re} z < 2\}, \dots, X_n = \{z : n-1 < \operatorname{Re} z\},$$

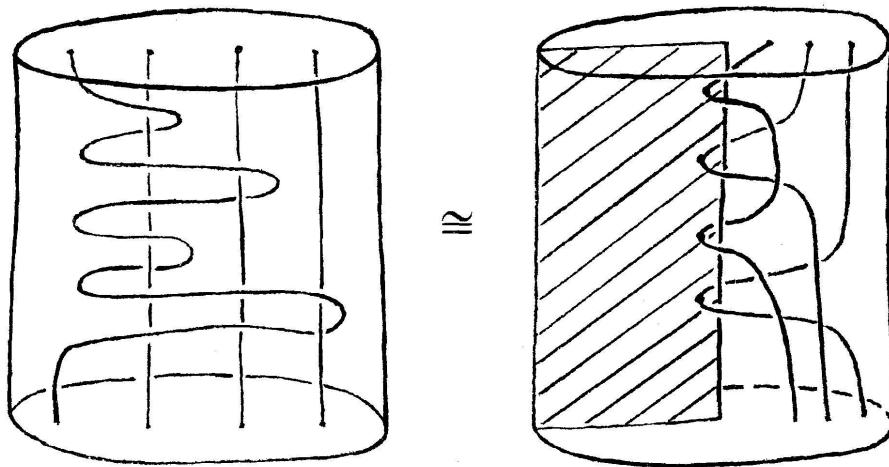
known in the classical style as *sheets* of the branched cover. Now if we consider $E_n - \Delta$ to be the configuration space of X , the inverse of the covering map defines a continuous function from $\mathbf{C} - \{1, \dots, n-1\}$ into $E_n - \Delta$, inducing a homomorphism from the free group $\pi_1(\mathbf{C} - \{1, \dots, n-1\}; 0)$ to the braid group $\pi_1(E_n - \Delta; p^{-1}(0))$. One readily checks that this homomorphism is onto, carrying the obvious free generator x_j of the free group (Figure 2) to the standard generator $\sigma_j \in B_n$. Let $v = x_{j(1)} \dots x_{j(k)}$ be any strictly positive word in

$$x_1, \dots, x_{n-1}, \beta = \sigma_{j(1)} \dots \sigma_{j(k)} = (p^{-1})_*(v)$$

FIGURE 2 ($n=4$)

$$(p^{-1})_*(x_2) = \sigma_2$$

its strictly positive image in B_n . We use v to construct an auxiliary closed braid in $S^1 \times \mathbf{C}$, the closure of $v' = A_{1, j(1)} \dots A_{1, j(k)} \in B_{n+1}$, where $A_{1, j} = (\sigma_1 \dots \sigma_{j-1})\sigma_j^2(\sigma_1 \dots \sigma_{j-1})^{-1}$ is one of the standard generators $A_{i, j}$ of the *pure braid group* (cf. [Bi] or see below). Now, v' can be realized as a geometric braid in two special ways: the first string can be made to wind in and out among the others, which are all straight; or the first string may be made straight, while the others wind around it in a succession of loops (Figure 3). On the first

FIGURE 3 ($n=4$)

$$v = x_1 x_2 x_1 x_3$$

interpretation, identifying the straight strings with $[0, 2\pi] \times \{1, \dots, n-1\}$, the winding first string becomes the graph of a loop

$$l: ([0, 2\pi], \{0, 2\pi\}) \rightarrow (\mathbf{C} - \{1, \dots, n-1\}, 0)$$

in the homotopy class v ; and its inverse image under the branched covering $id_{S^1} \times p: S^1 \times X \rightarrow S^1 \times \mathbf{C}$ is a geometric braid representing β . On the second interpretation, identifying the single straight string with $[0, 2\pi] \times \{0\}$, and taking care that each other string winds monotonically around this axis, the fibration of $S^1 \times (\mathbf{C} - \{0\})$ over S^1 by $(e^{i\theta}, w) \mapsto \arg w$ lifts back through the branched covering to a fibration of $(S^1 \times X) - \hat{\beta}$ over S^1 . (The strictness is used at this point, to ensure that in fact there is a non-zero winding number for each string. Positivity, however, could be weakened to “homogeneity” in the sense of [St].) There is no trouble “at infinity”, so that the fibration can be extended over all of $S^3 - \hat{\beta}$. Note that the fibre surface for $\hat{\beta}$ is the union of n disks with a surface that is the cover of an annulus branched at $e(\beta)$ points, so it has Euler

characteristic $n - e(\beta)$ and hence (being connected) genus $g = 1 - \frac{1}{2}(n - e(\beta) + c)$ if $\hat{\beta}$ has c components. This is the same genus formula as before when the link of a singularity is considered.

Besides exponent sum, there are other representations of B_n with applications here. First recall the *permutation representation* $\pi: B_n \rightarrow \mathcal{S}_n$, which takes σ_j to $(j \ j+1)$, $j = 1, \dots, n-1$. The kernel $\ker \pi$ is the group of *pure braids*; it is the fundamental group of the space of *ordered* n -tuples of distinct complex numbers. Let S_n be the free abelian group of rank $\frac{1}{2}n(n-1)$ consisting of symmetric n -by- n integer matrices with 0 diagonal. Now, in general, a cycle in $\pi(\beta)$ corresponds to a component of $\hat{\beta}$; and in particular the closure of a pure braid consists of n (unknotted) components which are naturally ordered $1, \dots, n$. Define $\lambda: \ker \pi \rightarrow S_n$ by setting $\lambda(\beta)_{i,j}$ equal to twice the linking number of the i -th and j -th components of $\hat{\beta}$, for β pure. These representations are combined in $\omega: B_n \rightarrow S_n \ltimes \mathcal{S}_n$, where in the semidirect product \mathcal{S}_n acts on S_n by conjugation with the standard permutation matrices, and

$$\omega(\sigma_i) = ([\delta_{i,i+1} + \delta_{i+1,i}], (i \ i+1)), \quad i = 1, \dots, n-1.$$

Let \mathcal{S}_n act diagonally on $\{1, \dots, n\}^2$, and let $|x| \cdot (i, j)$ denote the orbit of (the cyclic subgroup generated by) $x \in \mathcal{S}_n$ on (i, j) . Then for

$$i \neq j, \beta \in B_n, \omega(\beta) = ([a_{pq}], x),$$

the sum $\sum_{(p, q) \in |x| \cdot (i, j)} a_{pq}$ is an integer invariant of β , and appropriate sums of such invariants are conjugacy class invariants. In particular, when $\pi(\beta)$ is an n -cycle (so that $\hat{\beta}$ is a knot), such a conjugacy class invariant arises by summing over pairs (i, j) with a fixed constant difference modulo n : and this may be seen to be precisely twice one of the *self-windings* sw_i introduced by Laufer [Lau]. Laufer showed that the sw_i ($i = 1, \dots, n$) suffice to distinguish the knot types of links of unibranch singularities; in fact, he showed that the Puiseux pairs of a branch could be reconstructed from the self-windings. Simple examples show that $sw = e$ and the sw_i (and even their slight generalizations just given) can't tell apart all quasipositive, or even all positive, closed braids. It is interesting to speculate that there might be reasonable representations λ_1 of $\ker \lambda$, λ_2 of $\ker \lambda_1, \dots$, which could somehow be combined into a (faithful?) representation of B_n in which quasipositivity might show up more clearly than it does in B_n itself. (Is there any relation to Laufer's other numerical link invariants [Lau 2]? Perhaps λ_1 can be constructed out of linking numbers in branched covers of S^3 , branched over one

of the—unknotted!—components of a pure link in which every linking number is 0; and so on.)

As a final topic, we return to “knot groups” of plane curves and related matters, from a braid-theoretical point of view.

As before, let R be the compact region of \mathbf{C} bounded by a simple closed curve γ . Let S be a compact oriented surface-with-boundary. Then a map $f : S \rightarrow R \times \mathbf{C}$, or its image $f(S)$, is a *braided surface* of degree $n \geq 1$ provided that $pr_1 \circ f : S \rightarrow R$ is a branched covering of degree n : f is a *smooth*, *analytic*, or *algebraic* braided surface if $f(S)$ is smooth, complex analytic, or (complex) algebraic. Let $V'_f \subset S$ and $V_f \subset R$ denote the branch sets of $pr_1 \circ f$, finite sets avoiding ∂S and γ ; and let $W_f, V_f \subset W_f \subset R$, be the set $\{z \in R : (\{z\} \times \mathbf{C}) \cap f(S) \text{ contains fewer than } n \text{ points}\}$. One can interpret f^{-1} as a map, as smooth as f , from R into E_n . As remarked earlier, when f^{-1} is transverse to Δ , then $W_f = V_f$ and f is a smooth braided surface; but f can be smooth without f^{-1} being transverse to Δ . (Consider non-generic “vertical” tangencies.) Nor need W_f be finite, *but we will always assume that it is*, even when f^{-1} is not transverse to Δ . With this proviso, *every braided surface* f is a topological (even p.l.) immersion, though not necessarily locally flat. To see this, define the *local braid of f at $z \in R$* , denoted $\beta_{f,z} \in B_n$, to be the homotopy class of the loop $\theta \rightarrow f^{-1}(z + \varepsilon e^{i\theta})$, $0 \leq \theta \leq 2\pi$, for any sufficiently small $\varepsilon > 0$. (Since the basepoints of the various copies of B_n vary with z , $\beta_{f,z}$ is really only defined up to conjugacy.) This is well-defined when W_f is finite (or even as long as z is not an accumulation point of W_f); of course $\beta_{f,z} = 1$ if and only if $z \in R - W_f$. For $z \in W_f$, $\beta_{f,z}$ has strictly fewer than n components, which will be grouped into possibly yet fewer unsplittable links. Then $f(S)$, above z , is embedded in $R \times \mathbf{C}$ like disjoint cones (with distinct vertices) on the unsplittable sublinks of $\hat{\beta}_{f,z}$. For example, if $z \in V_f$ lies under only a simple vertical tangent, then $\beta_{f,z}$ is a band (positive or negative), which might as well be taken to be $\sigma_1^{\pm 1} \in B_n$, and $\hat{\beta}_{f,z}$ is a split link of $n - 1$ unknotted components.

Recall (cf. [Bi]) that B_n acts (faithfully) as a group of automorphisms of the free group F_n of rank n . Explicitly, if $F_n = \pi_1(\mathbf{C} - \{w_1, \dots, w_n\}; w_0)$, the acting B_n is realized as $\pi_1(E_n - \Delta; \{w_1, \dots, w_n\})$; on standard free generators x_1, \dots, x_n of F_n (positively oriented meridians around w_1, \dots, w_n), the action is

$$x_i \sigma_i = x_i x_{i+1} x_i^{-1}, \quad x_{i+1} \sigma_i = x_i, \quad x_j \sigma_i = x_j$$

for $j \neq i, i + 1$. Pick a basepoint $z_0 \in R - W_f$, and paths from z_0 to the points z_1, \dots, z_k of W_f . By these paths, all the local braids can be taken to lie in one and the same braid group, namely, $\pi_1(E_n - \Delta; pr_2((\{z_p\} \times \mathbf{C}) \cap f(S)))$ —denote by $\beta'_{f,z}$ these braids. (Simple vertical tangents, for instance, will now give braids $\beta'_{f,z}$

which are bands that *cannot* all at once be taken to be $\sigma_1^{\pm 1}$.) It may now be seen that

$$(x_1, \dots, x_n : x_i \beta'_{f,z} = x_i, i = 1, \dots, n, z \in W_f)$$

is a presentation of the “knot group” $\pi_1((R \times \mathbf{C}) - f(S); *)$. When f is algebraic and γ is a very large circle this is really van Kampen’s presentation (except for the relation “at infinity” to which we will return shortly).

A finite presentation of a group, in which each relation sets one generator equal to some conjugate of another generator, is a *Wirtinger presentation*; a group with a Wirtinger presentation is a *Wirtinger group*. Any Wirtinger group has a *simple* Wirtinger presentation, in which each relation is of the form $x_i x_j x_i^{-1} = x_k$, for not necessarily distinct generators x_i, x_j, x_k . After possibly adding more generators, and renumbering them, one can assume that each relation is of one of the two forms $x_i = x_{j+1}$ or $x_i = x_j x_{j+1} x_j^{-1}$, $i < j$. These two relations are contributed, respectively, by the action on F_n of

$$(\sigma_i \sigma_{i+1} \dots \sigma_{j-1}^\varepsilon) \sigma_j (\sigma_i \sigma_{i+1} \dots \sigma_{j-1}^\varepsilon)^{-1}, \varepsilon = +1 \text{ or } -1.$$

So every Wirtinger group has a simple Wirtinger presentation which is the van Kampen presentation of the fundamental group $\pi_1(\{(z, w) \in \mathbf{C}^2 : |z| \leq 1\} - f(S); *)$ for some smooth braided surface $f(S)$ with boundary the closure of a quasipositive braid (the product of the bands used to achieve the desired relations); and actually $f(S)$ can be taken to be non-singular complex analytic. So we see that *the class of knot groups of complex analytic curves in a bidisk is exactly the class of Wirtinger groups*, a refinement [Ru 2] of results of Yajima [Ya] and Johnson [Jo] (who weren’t concerned with complex analytic structures).

If one wishes to investigate knot groups for smooth braided surfaces of fixed topological type, one still loses nothing by demanding that the surfaces be complex curves: if $f(S)$ is smooth, by slight jiggling f^{-1} becomes transverse to Δ while $f(S)$ moves by an isotopy; then the braids $\beta'_{f,z}$ are all bands, positive or negative; changing all the signs to positive reembeds S as a quasipositive braided surface, and therefore, up to isotopy, a complex analytic curve; but it does not change the knot group at all, since $x\beta^{-1} = x$ is the same relation as $x = x\beta$.

So far, everything has been phrased for braided surfaces over a compact (simply-connected) region R . If we replace R by all of \mathbf{C} , much stays the same; it is now appropriate to let W_f be infinite, but discrete. It ceases to be clear, however, (at least to this author at the present time) that a quasipositive “infinite band representation” can always be realized by an entire n -valued analytic function. Also, as observed in [Ru 1], for compact R , at least as far as the boundary closed

braid is concerned, every n -valued analytic function can be assumed to be the restriction of an entire n -valued *algebraic* function; this is certainly not true for $R = \mathbf{C}$, because the “local braid at infinity” $\beta_{f,\infty}$ of an algebraic braided surface over \mathbf{C} —i.e., the braid over a simple closed curve large enough to enclose V_f entirely—is severely restricted. Its closure, for instance, is an iterated torus link (as we saw in the proof of the theorem of Abhyankar and Moh, § 6). And if the projective completion of the algebraic braided surface (algebraic curve), in \mathbf{CP}^2 , meets the line at infinity transversely, one actually has $\hat{\beta}_{f,\infty}$ the union of n circles of the Hopf fibration $S^3 \rightarrow \mathbf{CP}^1$ —the braid $\beta_{f,\infty}$ is the generator of the (infinite cyclic) center of $B_n (n \geq 3)$, which bears the name Δ^2 (unfortunately, in this context), cf. [Bi]. Any knot group of a projective plane curve, then, can be presented by starting with an expression of Δ^2 as a product $\beta(l) \cdots \beta(k)$ in B_n , where each $\beta(i)$ is conjugate in B_n to some local braid associated to the link of a singularity (including non-trivial local braids which are associated to the unknotted link of a regular point!), then forming the presentation

$$(x_1, \dots, x_n : x_1 x_2 \cdots x_n = 1, x_i \beta(j) = x_i, i = 1, \dots, n, j = 1, \dots, k).$$

For instance, a quasipositive band representation of Δ^2 (each $\beta(i)$ a positive band, that is, conjugate to the nontrivial local braid σ_1 associated to a simple vertical tangent) corresponds to a non-singular curve of degree n , and presents $\mathbf{Z}/n\mathbf{Z}$. A quasipositive *nodal* band representation, where each $\beta(i)$ is either a positive band or the square of a positive band, corresponds to a node curve; if some $\beta(i)$ are cubes of positive bands, others squares or first powers, we have a *cuspidal* band representation; and so on. There is a mapping from the set of strata of Q_n (§ 5) into a hierarchy of “types of expressions” of $\Delta^2 \in B_n$ as products $\beta(l) \cdots \beta(k)$; Moishezon’s problem of normal forms is a first step in the study of this mapping, about which little seems to be known. Is it onto? An affirmative answer would be a strong generalization of Riemann’s Existence Theorem. (Again, cf. [Mo].)

We conclude with three examples. First recall some formulas for Δ^2 in B_n (cf. [Bi] or [Mo]): $\Delta^2 = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n$; also, Δ^2 is pure, and in terms of the standard generators

$$A_{ij} = (\sigma_i \cdots \sigma_{j-1}) \sigma_j^2 (\sigma_i \cdots \sigma_{j-1})^{-1}, \quad 1 \leq i \leq j \leq n-1,$$

of the pure braid group,

$$\Delta^2 = A_{1,n-1} A_{1,n-2} \cdots A_{1,1} A_{2,n-1} \cdots A_{2,2} \cdots A_{n-1,n-1}.$$

Example 1. Write $\Delta^2 = \beta(l) \cdots \beta(n^2 - n)$, $\beta(i) = \beta_{i \bmod n-1}$, as just given. It is easy to see that this expression for Δ^2 does in fact correspond to a non-singular

curve of degree n . The corresponding presentation of the knot group of the curve includes among its relations $x_1 x_2 \cdots x_n = 1$ and each equality $x_i = x_{i+1}$, $i = 1, \dots, n-1$. So the knot group is a quotient of $\mathbf{Z}/n\mathbf{Z}$; but a simple homological argument shows that $\mathbf{Z}/n\mathbf{Z}$ is the abelianization of the knot group, so the two groups are equal.

Example 2. Write $\Delta^2 = \beta(l) \cdots \beta((n^2 - n)/2)$, where $\beta(i) = A_{p,q}$ as above. Each pair (p, q) arises. The relations in the corresponding presentation say that for each pair p, q the generators x_p, x_{q+1} commute. (For instance, the action of

$$A_{1,1} = \sigma_1^2 \text{ on } F_n \text{ is } x_1 \sigma_1^2 = (x_1 x_2 x_1^{-1}) \sigma_1 = x_1 x_2 x_1 x_2^{-1} x_1^{-1}, \\ x_2 \sigma_1^2 = x_1 \sigma_1 = x_1 x_2 x_1^{-1}, x_k \sigma_1 = x_k, k \neq 1, 2;$$

and the relations $x_1 = x_1 x_2 x_1 x_2^{-1} x_1^{-1}$ and $x_2 = x_1 x_2 x_1^{-1}$ both say x_1 commutes with x_2 .) The group is free abelian of rank $n-1$. Moishezon sketches a proof that this presentation does arise geometrically; another proof could be given by the method of [Ru 1].

Example 3. For $n = 4$, $\Delta^2 = \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_3 \sigma_1 \sigma_2 \sigma_3$. Let us suppress the symbol σ , raise subscripts (so k denotes σ_k), and write, for instance, $\overline{2}^3 4$ to mean $\sigma_2 \sigma_3^{-1} \sigma_4 \sigma_3 \sigma_2^{-1}$. Then, by dogged manipulation, $\Delta^2 \in B_4$ can be worked into the form $(3 \cdot 3 \cdot 3) (\overline{3}^2 1) (1 \cdot 1 \cdot 1) (2) (1 \cdot 1 \cdot 1) (\overline{3}^2 1)$, the product of three positive bands and three “cusps” (cubes of positive bands). The corresponding presentation, before adjoining the relation at infinity, presents the group of the 5-twist spun trefoil (as has been remarked by Dewitt Sumners); with that relation, $x_1 x_2 x_3 x_4 = 1$, the group becomes the non-abelian group of order 12, $(a, b : aba = bab, a^4 = 1, a^2 = b^2)$. This is the correct group [Z] for a tricuspidal cubic curve, and presumably the given “quasipositive cuspidal band representation” really arises geometrically, but I have not had the courage to check this.—Similarly, for $n = 6$, $\Delta = 123451234123121$, which can be written as $(1 \cdot 1 \cdot 1) (\overline{1}^2 1) (3 \cdot 3 \cdot 3) (\overline{3}^4 3) (5 \cdot 5 \cdot 5) (\overline{1}^3 3) 2 (\overline{3}^5 5) 4 (2^3) (4^5)$; the presentation for the square of this, with the relation at infinity, is at an intermediate stage $(x_1, x_2, x_3, x_4, x_5, x_6 : x_1 = x_3 = x_5, x_2 = x_4 = x_6, x_1 x_2 x_1 = x_2 x_1 x_2, x_1 x_2 x_3 x_4 x_5 x_6 = 1)$ which becomes $(a, b : a^2 = b^3 = 1)$, the group given in [Z] for a sextic with six cusps on a conic. On the other hand, a less symmetrical way to write $\Delta^2 \in B_6$ is as $(\overline{2}^1 2 3) (4) (5) (2 \cdot 2 \cdot 2)^1 (2 \cdot 2 \cdot 2) (\overline{3}^2 2) (4^3 1 2) (1 \cdot 1 \cdot 1) (\overline{4}^3 2 1) (4^4 5) \cdot (4^4 2 3) (4 \cdot 4 \cdot 4) (1 \cdot 1 \cdot 1) (\overline{2}^1 2 3)^2 (1 \cdot 1 \cdot 1) (\overline{2}^2 1) (2)$, which presents the abelian group $\mathbf{Z}/6\mathbf{Z}$ which [Z] gives for a sextic with six cusps *not* all on the same conic.

REFERENCES

- [Abe] ABELSON, Harold. Fundamental groups of plane curves and their duals. *Indiana Univ. Math. J.* 25 (1976), No. 1, 65-67. MR 53 # 5603.
- [Ab] ABHYANKAR, Shreeram. Tame coverings and fundamental groups of algebraic varieties. V. Three cuspidal plane quartics. *Amer. J. Math.* 82 (1960), 365-373. MR 22 # 1577.
- [A-M] ABHYANKAR, S. and Tzuong-Tzieng MOH. Embeddings of the line in the plane. *J. reine ange. Math.* 276 (1975), 148-166. MR 52 # 407.
- [A-S] ABHYANKAR, S. and Balwant SINGH. Embeddings of certain curves in the affine plane. *Amer. J. Math.* 100 (1978), 99-175. MR 58 # 16663.
- [Bi] BIRMAN, Joan. *Braids, Links, and Mapping Class Groups*. Ann. Math. Studies 82 (1975), Princeton University Press.
- [Bi-W] BIRMAN, J. and R. F. WILLIAMS. Knotted Periodic Orbits in Dynamical Systems. I: Lorenz's Equations. *Topology* 22 (1983), 47-82.
- [Bl] BLISS, Gilbert Ames. *Algebraic Functions*. Amer. Math. Soc. Colloquium Publications, vol. XVI (1933).
- [Cha] CHANG, Haichau. On two classical facts by Zariski and van Kampen. *Chinese J. Math.* 7 (1979), 153-161.
- [Che] CHENIOT, D. Une démonstration du théorème de Zariski sur les sections hyperplanes d'une hypersurface projective et du théorème de van Kampen sur le groupe fondamental du complémentaire d'une courbe projective plane. *Compositio Math.* 27 (1973), 141-158.
- [De] DELIGNE, P. Le groupe fondamental du complément d'une courbe plane n'ayant que des points doubles ordinaires est abélien. *Sém. Bourbaki*, Nov. 1979.
- [E-N] EISENBUD, D. and Walter NEUMANN. Fibering Iterated Torus Links. (*To appear*).
- [F-H] FULTON, W. and J. HANSEN. A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings. *Ann. Math.* 110 (1979), 159-166.
- [G-R] GUNNING, R. C. and Hugo ROSSI. *Analytic Functions of Several Complex Variables*. Prentice-Hall, 1965.
- [H] HITOTUMATU, Sin. Some recent topics in several complex variables by the Japanese school. (Report of work by T. Nishino), in *Proceedings of the Romanian-Finnish Seminar on Teichmüller Spaces and Quasi-Conformal Mappings*, Brasov, Romania, 1969, Acad. Soc. Rep. Romania (1971), 187-191.
- [Jo] JOHNSON, D. *Homomorphs of knot groups*. Preprint, Jet Propulsion Laboratory, Pasadena, California (1978).
- [Lau] LAUFER, Henry B. On the number of singularities of an analytic curve. *Trans. Amer. Math. Soc.* 136 (1969), 527-535.
- [Lau 2] —— Some numerical link invariants. *Topology* 10 (1971), 119-131.
- [Lê] LÊ DUNG TRANG. Sur les nœuds algébriques. *Compositio Math.* 25 (1972), 281-321.
- [Li] LITHERLAND, R. A. Signatures of iterated torus knots. In *Topology of low-dimensional manifolds*, Lecture Notes in Mathematics 722 (1979), Springer, Berlin, 71-84. MR # 80k-57012.
- [Mi 1] MILNOR, J. W. *Morse Theory*. Ann. Math. Studies 51 (1965), Princeton University Press.

- [Mi 2] —— *Singular Points of Complex Hypersurfaces*. Ann. Math. Studies 61 (1969), Princeton University Press.
- [Mo] MOISHEZON, B. G. Stable branch curves and braid monodromies. *Algebraic Geometry (Chicago, Illinois, 1980)*, Lecture Notes in Mathematics 862 (1981), 107-192, Springer, Berlin.
- [Ru 1] RUDOLPH, Lee. Algebraic functions and closed braids. *Topology* 22 (1983), 191-202.
- [Ru 2] —— Braided surfaces and Seifert ribbons for closed braids. To appear in *Comm. Math. Helv.* (1983).
- [Ru 3] —— Constructions of quasipositive knots and links, I. In *Nœuds, tresses et singularités*, Monographie de l'Enseignement Mathématique N° 31, Genève 1983, 233-245.
- [Ru 4] —— Embeddings of the line in the plane. *J. reine ange. Math.* 337 (1982), 113-118.
- [Ru 5] —— Non-trivial positive braids have positive signature. *Topology* 21 (1982), 325-327.
- [Ru 6] —— Question printed in “Queries” column. *Notices Amer. Math. Soc.* 23 (1976), p. 410 (from a problem list compiled at the Special Session on Knot Theory, 1976 Summer Meeting of A.M.S., Toronto).
- [St] STALLINGS, J. Constructions of fibred knots and links. In *Algebraic and Geometric Topology* (Proc. Symp. Pure Math. XXXII), part 2 (1978), 55-60, Amer. Math. Soc., Providence, R. I.
- [vK] VAN KAMPEN, E. R. On the fundamental group of an algebraic curve. *Amer. J. Math.* 55 (1933), 255-260.
- [Ya] YAJIMA, T. Wirtinger presentations of knot groups. *Proc. Jap. Acad. Sci.* 46 (1970), 997-1000.
- [Z] ZARISKI, O. On the problem of existence of algebraic functions of two variables possessing a given branch curve. *Amer. J. Math.* 51 (1929), 305-328.

(Reçu le 30 septembre 1982)

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