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§3. Compact groups. Proof of Theorem A.

1. Let U be a compact Lie group. Then we may view U as the group $G(\mathbf{R})$ of real points of an algebraic group G defined over $\mathbf{R}$ [5]. Furthermore, the maximal (topological) tori of U are the groups $T(\mathbf{R})$, where T runs through the maximal $\mathbf{R}$-tori of G. They are conjugate under inner automorphisms of U. Corollary 1 to Theorem 2 insures the existence of a non-commutative free subgroup Γ of U such that every $\gamma \in \Gamma - \{1\}$ is strongly regular, i.e., generates a dense subgroup of a maximal torus of U. If now V is a closed subgroup of U, then, by [10], $\chi(U/V) = 0$ if V does not contain a maximal torus of U, and is equal to $[N_U(T) : N_V(T)]$ if V contains a maximal torus T of U. By the results just recalled, we may write $V = H(\mathbf{R})$, where H is an algebraic $\mathbf{R}$-subgroup of G, the condition (*) of §2 is satisfied, and any maximal torus of U is conjugate to T. Theorem A now follows from Corollaries 1 and 3 to Theorem 2.

2. The results of this paper, specialized to compact Lie groups, can of course be proved more directly, in the framework of the theory of compact Lie groups, without recourse to the theory of algebraic groups. For the benefit of the reader mainly interested in that case, we sketch how to modify the above arguments.

The main point is again to prove Theorem 1, where now G stands for a non-trivial compact connected semi-simple Lie group. In part a) of the proof, the role of $\mathbf{SL}_n$ is taken by $\mathbf{SU}_n$. If $n = 2$, G contains non-commutative free subgroups. If $n > 2$, the argument is the same except that now we take for D, exactly as in [8], a division algebra with an involution of the second kind and identify $\mathbf{SU}_n$ to $(D \otimes_L \mathbf{R})^1$, where L is the fixed field, in the center of D, of the given involution of D. In part b), we use the fact that if G is simple, not locally isomorphic to $\mathbf{SU}_n$, then it contains a proper closed connected semi-simple subgroup of maximal rank, for which we can refer directly to [2] (the proof of Lemma 1 was in fact just an adaptation to algebraic groups of the one in [2]).

Then, as pointed out in section 5 of §2, a simple category argument yields Theorem 2, whence also Corollary 1 to Theorem 2 and Theorem A.

§4. Free group actions with commutative isotropy groups

1. Let Γ be a non-commutative free group acting on a set X. Assume that Γ acts freely, or more generally, that the isotropy groups $\Gamma_x (x \in X)$ are commutative (hence cyclic), and that at least one is reduced to $\{1\}$. Then the decomposition theorem 2.2.1, 2.2.2 of [6] implies in particular the following: given $n \geqq 2$, there exists a partition of X into $2n$ subsets X_i and elements $\gamma_i \in \Gamma (1 \leqq i \leqq 2n)$ such that X is the disjoint union of $\gamma_i X_i$ and $\gamma_{n+i} X_{n+i} (i \leqq i \leqq n)$. If we view the operations of

Γ as congruences, this shows that X is equivalent to the union of n copies of itself via finite congruences. The existence of such partitions of $\mathbf{S}^2$ was proved first by R. M. Robinson [13].

This then leads to the problem of finding actions of free groups with commutative isotropy groups in cases where free actions are ruled out. We now prove some results pertaining to that question.

2. Consider first the case of $\mathbf{S}^n = \mathbf{SO}_{n+1}/\mathbf{SO}_n$. The problem is then to find a free non-commutative subgroup Γ of $\mathbf{SO}_{n+1}$ such that no two non-commutative elements of Γ are contained in a conjugate of $\mathbf{SO}_n$, i.e., have a common non-zero fixed vector. In [6], this is shown for $n \geqq 2$, but $n \neq 4$. We want to give an alternate proof which also covers that last case. For n odd, there is even a Γ such that no element $\neq 1$ has an eigenvector, as follows from the remark to Theorem 2. So assume n even. If $n = 2$, then the isotropy groups of $\mathbf{SO}_3$ itself on $\mathbf{S}^2$ are commutative, hence any non-commutative free subgroup will do. Assume $n > 2$. The group $\mathbf{SO}_3$ has an (absolutely) irreducible real representation of degree $n + 1$; it can e.g. be realized in the space of spherical harmonics in $\mathbf{R}^3$ of degree $n/2$. Let H be the image of $\mathbf{SO}_3$ in $\mathbf{SO}_{n+1}$ under such a representation and let Γ be a free non-commutative subgroup of H. Then any two non-commuting elements of Γ generate a dense subgroup of H, hence do not have a common non-zero proper invariant subspace of $\mathbf{R}^{n+1}$; in particular they have no common fixed vector, whence our assertion.

Example. For the sake of definiteness, we indicate one explicit example in the case $n = 4$.

Let $\alpha, \beta \in (0, 2\pi)$ be two angles such that the rotations of angle α and β of $\mathbf{R}^3$ around two perpendicular axes freely generate a free subgroup $F_{\alpha,\beta}$ of $\mathbf{SO}_3$. We may take e.g. $\alpha = \beta$, where α is such that $\cos\alpha$ is transcendental [7]. Let $\{e_1, \ldots, e_5\}$ be the canonical basis of $\mathbf{R}^5$. Let $A_\alpha \in \mathbf{SO}_5$ be the transformation which is a rotation of angle 2α in the plane $[e_4, e_5]$ spanned by e_4 and e_5 and which is the rotation of angle 4α around the axis spanned by $(3^{1/2}, 0, 1)$ in $[e_1, e_2, e_3]$. Let B_β the element of $\mathbf{SO}_5$ which fixes e_3 and is a rotation of angle 2β (resp. 4β) in the plane $[e_2, e_4]$ (resp. $[e_1, e_5]$). Then A_α and B_β freely generate an irreducible subgroup of $\mathbf{SO}_5$, whose closure is isomorphic to $\mathbf{SO}_3$ and which is therefore locally commutative on $\mathbf{S}^4$.

In fact, in suitable coordinates, this group is just the image of the group $F_{\alpha,\beta}$ under the five-dimensional irreducible representation of $\mathbf{SO}_3$. The easy computations showing this are left to the reader.

3. The above argument extends in the general case to the following sharpening of Theorem A in the case of non-zero Euler characteristic.

THEOREM 3. *Let* U *be a compact connected non-trivial semi-simple Lie group. Then* U *contains a non-commutative free subgroup* Γ *whose elements* $\gamma \neq 1$ *are regular and such that, for any proper closed subgroup* V *of maximal rank of* U, *the isotropy groups* $\Gamma_x (x \in U/V)$ *of* Γ *on* U/V *are commutative and any* $\gamma \in \Gamma - \{1\}$ *has exactly* $\chi(U/V)$ *fixed points.*

Proof: First we carry an easy reduction to the case where U is simple and V connected. Let U' be the quotient of U by its center, $\pi : U \to U'$ the natural projection and $V' = \pi(V)$. The isotropy groups of U on U'/V' contain the isotropy groups on U/V, hence we may assume that U has center reduced to the identity. Let V^0 be the identity component of V. Any isotropy group of Γ on U/V contains an isotropy group on U/V^0 as a subgroup of finite index. Both are therefore simultaneously commutative or not commutative. So we may assume V to be connected. Now U is a direct product of simple groups and V, being of maximal rank, is the direct product of its intersections with the simple factors of U [2], whence our reduction.

We now prove the theorem in this case except for the last assertion on the number of fixed points.

If $U = \mathbf{SO}_3$, then any proper closed subgroup has a commutative subgroup of finite index, and any element $\neq 1$ is regular. Therefore we may take for Γ any non-commutative free subgroup. Assume now that $U \neq \mathbf{SO}_3$, hence $\dim U > 3$. Then U has a closed subgroup H, isomorphic to $\mathbf{SO}_3$, which contains regular elements of U and is not contained in any proper subgroup of maximal rank [15: §12]. (This subgroup is called "principal" in [15].) Then any element of infinite order in H is regular in U. In particular any element $\gamma \neq 1$ in a free non-commutative subgroup Γ of H is regular. Moreover any two non-commuting elements of Γ generate a dense subgroup of H. If they were contained in a conjugate of V, then so would H, whence a contradiction.

There remains to see that every $\gamma \in \Gamma - \{1\}$ has exactly $\chi(U/V)$ fixed points on U/V. Let S_γ be the closure of the subgroup of H generated by γ. It is a one-dimensional torus, almost all of whose elements are regular in U. Fix a maximal torus T_0 of V, hence of U. If $x, y \in U$ are such that ${}^xS, {}^yS \subset T_0$, then the inner automorphism by $x \cdot y^{-1}$, which brings ${}^yS_\gamma$ onto ${}^xS_\gamma$, must leave T_0 stable since ${}^xS_\gamma$ contains regular elements, i.e., $x \cdot y^{-1} \in N_U(T_0)$. From this we see again that there is a natural bijection between the fixed point set of γ and $N_U(T_0)/N_V(T_0)$, and our assertion follows as in section 4 of §2.

4. The same argument is valid for a complex semi-simple Lie group, using a principal three-dimensional subgroup, or also over any algebraically closed groundfield. Over a field K of infinite transcendence degree over its prime field,

one would have to assume the existence of a principal three-dimensional subgroup which is defined over K.

5. We note finally that if $\Gamma \subset G(K)$ satisfies the conditions of Corollary 1 to Theorem 2 and if H is a subgroup of maximal rank of G whose identity component is solvable, then for any $x \in G(K)/H(K)$, the isotropy group Γ_x is commutative, since its intersection with the isotropy group of x in $G(K)$ is on one hand free, as a subgroup of Γ, and on the other hand contains a solvable normal subgroup of finite index, since $H(K)$ does.

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