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§3. Compact groups. Proof of Theorem A.

1. Let U be a compact Lie group. Then we may view U as the group $G(\mathbf{R})$ of real points of an algebraic group G defined over $\mathbf{R}$ [5]. Furthermore, the maximal (topological) tori of U are the groups $T(\mathbf{R})$, where T runs through the maximal $\mathbf{R}$-tori of G. They are conjugate under inner automorphisms of U. Corollary 1 to Theorem 2 insures the existence of a non-commutative free subgroup Γ of U such that every $\gamma \in \Gamma - \{1\}$ is strongly regular, i.e., generates a dense subgroup of a maximal torus of U. If now V is a closed subgroup of U, then, by [10], $\chi(U/V) = 0$ if V does not contain a maximal torus of U, and is equal to $[N_U(T) : N_V(T)]$ if V contains a maximal torus T of U. By the results just recalled, we may write $V = H(\mathbf{R})$, where H is an algebraic $\mathbf{R}$-subgroup of G, the condition (*) of §2 is satisfied, and any maximal torus of U is conjugate to T. Theorem A now follows from Corollaries 1 and 3 to Theorem 2.

2. The results of this paper, specialized to compact Lie groups, can of course be proved more directly, in the framework of the theory of compact Lie groups, without recourse to the theory of algebraic groups. For the benefit of the reader mainly interested in that case, we sketch how to modify the above arguments.

The main point is again to prove Theorem 1, where now G stands for a non-trivial compact connected semi-simple Lie group. In part a) of the proof, the role of $\mathbf{SL}_n$ is taken by $\mathbf{SU}_n$. If $n = 2$, G contains non-commutative free subgroups. If $n > 2$, the argument is the same except that now we take for D, exactly as in [8], a division algebra with an involution of the second kind and identify $\mathbf{SU}_n$ to $(D \otimes_L \mathbf{R})^1$, where L is the fixed field, in the center of D, of the given involution of D. In part b), we use the fact that if G is simple, not locally isomorphic to $\mathbf{SU}_n$, then it contains a proper closed connected semi-simple subgroup of maximal rank, for which we can refer directly to [2] (the proof of Lemma 1 was in fact just an adaptation to algebraic groups of the one in [2]).

Then, as pointed out in section 5 of §2, a simple category argument yields Theorem 2, whence also Corollary 1 to Theorem 2 and Theorem A.

§4. Free group actions with commutative isotropy groups

1. Let Γ be a non-commutative free group acting on a set X. Assume that Γ acts freely, or more generally, that the isotropy groups $\Gamma_x (x \in X)$ are commutative (hence cyclic), and that at least one is reduced to $\{1\}$. Then the decomposition theorem 2.2.1, 2.2.2 of [6] implies in particular the following: given $n \geqq 2$, there exists a partition of X into $2n$ subsets X_i and elements $\gamma_i \in \Gamma (1 \leqq i \leqq 2n)$ such that X is the disjoint union of $\gamma_i X_i$ and $\gamma_{n+i} X_{n+i} (i \leqq i \leqq n)$. If we view the operations of