

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 29 (1983)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ON FREE SUBGROUPS OF SEMI-SIMPLE GROUPS  
**Autor:** Borel, A.  
**Kapitel:** §2. Free subgroups with strongly regular elements  
**DOI:** <https://doi.org/10.5169/seals-52977>

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

### **Terms of use**

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

**Download PDF:** 18.04.2026

**ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>**

We need only to prove that  $\mathbf{SL}_2(\Omega)$  contains a non-commutative free subgroup  $F$ . If  $\Omega$  has characteristic zero, we may take any torsion-free subgroup of  $\mathbf{SL}_2(\mathbf{Z})$ . Let now  $p = \text{char } \Omega$  be  $> 0$ . Then, by the arithmetic method, using division quaternion algebras over global fields, we can construct a discrete cocompact subgroup of  $\mathbf{SL}_2(L)$ , where  $L$  is a local field of characteristic  $p$  (cf. A. Borel-G. Harder, *Crelle J.* 298 (1978), 53-74). The latter has a torsion-free subgroup  $F$  of finite index (H. Garland, *Annals of Math.* 97 (1973), 375-423) which is then free, since it acts freely on a tree, namely the Bruhat-Tits building of  $\mathbf{SL}_2(L)$ .

2) For any non-zero  $n \in \mathbf{Z}$ , the power map  $g \mapsto g^n$  is dominant (because it is surjective on any maximal torus [1 : 8.9]), hence Theorem 1 is obvious if the sum of the exponents of one letter in the word  $w$  is not zero. (See [11] for a similar remark in the context of compact groups.)

3) If  $U$  and  $V$  are non-empty open subsets in a connected algebraic group  $H$ , then  $H = U \cdot V$  [1 : 1.3]. It follows then from Theorem 1 that if  $w, w'$  are two words in two letters, say, then the map  $G^4 \rightarrow G$  defined by

$$f(g_1, g_2, g_3, g_4) = w(g_1, g_2) \cdot w'(g_3, g_4),$$

is surjective. For instance, every element of  $G(\Omega)$  is the product of two commutators. However, the map  $f_w$  itself is not always surjective; for instance  $x \mapsto x^2$  is not surjective in  $\mathbf{SL}_2(\mathbf{C})$ , as pointed out in [11].

4) If  $K = \mathbf{C}$ , then Theorem 1 implies that  $\text{Im } f_w$  contains a dense open set in the ordinary topology. If  $G$  is defined over  $\mathbf{R}$ , then Theorem 1 also shows that  $f_w(G(\mathbf{R}))$  contains a non-empty subset of  $G(\mathbf{R})$  which is open in the ordinary topology. However it may not be dense. For instance, it is pointed out in [11] that for  $\mathbf{SU}_2$ , the image of the map defined by  $[x^2, yxy^{-1}]$  omits a neighborhood of  $-1$ ; however this map is surjective in  $\mathbf{SO}_3$ .

It seems that little is known about the image of  $f_w$ , even over  $\mathbf{R}$  or  $\mathbf{C}$ . A general fact however is that the commutator map is surjective in any compact connected semi-simple Lie group [9].

## §2. FREE SUBGROUPS WITH STRONGLY REGULAR ELEMENTS

1. In the sequel,  $K$  is a field of infinite transcendence degree over its prime field. We shall need the following lemma:

LEMMA 2. *Let  $X$  be an irreducible unirational  $K$ -variety. Let  $L$  be a finitely generated subfield of  $K$  containing a field of definition of  $X$ , and  $V_i (i \in \mathbf{N})$  a sequence of proper irreducible algebraic subsets of  $X$  defined over an*

algebraic closure  $\bar{L}$  of  $L$ . Then  $X(K)$  is not contained in the union of the  $V_i \cap X(K)$ , ( $i \in \mathbf{N}$ ).

By definition of unirationality, there exists for some  $n \in \mathbf{N}$  a dominant  $K$ -morphism  $\varphi : \mathbf{A}^n \rightarrow X$ , where  $\mathbf{A}^n$  denotes the affine  $n$ -dimensional space.

This map is already defined over some finitely generated extension of  $L$ . Replacing  $L$  by the former, we may assume  $\varphi$  to be defined over  $L$ , hence  $\varphi^{-1}(V_i)$  to be defined over  $\bar{L}$ . It is a proper algebraic subset since  $\varphi$  is dominant. This reduces us to the case where  $X = \mathbf{A}^n$ . But then any point whose coordinates generate over  $\bar{L}$  a field of transcendence degree  $n$  will do.

**THEOREM 2.** *Assume  $G$  to be defined over  $K$ . Let  $\mathcal{V} = \{V_i\}$  ( $i \in \mathbf{N}$ ) be a family of proper subvarieties of  $G$ , all defined over an algebraic closure  $\bar{L}$  of a finitely generated subfield  $L$  of  $K$  over which  $G$  is also defined. Then  $G(K)$  contains a non-commutative free subgroup  $\Gamma$  such that no element of  $\Gamma - \{1\}$  is contained in any of the  $V_i$ 's. Given  $m \geq 2$ , the set of  $m$ -tuples which freely generate a subgroup having this property is Zariski dense in  $G^m$ .*

We may (and do) assume that the identity element is contained in one of the  $V_i$ 's.

Let  $w$  and  $f_w$  be as in §1. Then  $f_w$  is defined over  $L$  hence  $f_w^{-1}(Z)$  is defined over  $\bar{L}$  for every  $Z \in \mathcal{V}$  and is a proper algebraic subset by Theorem 1. The sets  $f_w^{-1}(Z)$ , as  $w$  runs through all the non-trivial reduced words (in  $m$  letters and their inverses) and  $Z$  through  $\mathcal{V}$ , form then a countable collection of proper algebraic subsets, all defined over  $\bar{L}$ . But  $G$ , hence  $G^m$ , is a unirational variety over any field of definition of  $G$  [1: 18.2]. Lemma 2 implies therefore the existence of  $g = (g_i) \in G(K)^m$  not belonging to any of these subsets. Then the  $g_i$ 's are free generators of a subgroup which satisfies our conditions. In fact, we see that we can take for  $g$  any point of  $G(K)^m$  which is generic over  $\bar{L}$  and, since  $\bar{L}$  has finite transcendence degree over the prime field, such points are Zariski-dense. This establishes the second assertion.

*Remark.* If  $G = \mathbf{SO}_{2n}$  (resp.  $\mathbf{SO}_{2n+1}$ ), this shows for instance the existence of a free subgroup  $\Gamma$ , no element of which except 1 has the eigenvalue 1 (resp. the eigenvalue 1 with multiplicity  $> 1$ ).

2. Any semi-simple element  $x$  of  $G$  is contained in a maximal torus [1: 11.10];  $x$  is called regular if it is contained in exactly one maximal torus. We shall say that  $x$  is *strongly regular* if it is not contained in any non-maximal torus, i.e., if the cyclic group generated by  $x$  is Zariski-dense in a maximal torus.

The following result contains Theorem C of the introduction.

COROLLARY 1. *Assume  $G$  to be defined over  $K$ . Then  $G(K)$  contains a non-commutative free subgroup  $\Gamma$  all of whose elements  $\neq 1$  are strongly regular. Given  $m \geq 2$ , the set of  $m$ -tuples  $(g_i) \in G(K)^m$  which generate freely a subgroup with that property is Zariski dense in  $G^m$ .*

The field  $K$  contains a field of definition  $L$  of  $G$  which is finitely generated over its prime field. Let  $\bar{L}$  be an algebraic closure of  $L$  in our universal field  $\Omega$ . Then the subfield generated by  $\bar{L}$  and  $K$  has infinite transcendence degree over  $\bar{L}$ . Let  $S$  be the set of singular elements of  $G$  (i.e., of elements  $g \in G$  such that  $\text{Ad } g$  has the eigenvalue one with multiplicity  $> \text{rk } G$ ). It is algebraic, defined over  $\bar{L}$ . Fix a maximal  $L$ -torus  $T$  of  $G$  [1: 18.2]. Every proper closed subgroup of  $T$  is contained in the kernel of a rational character [1: 8.2]. The characters are all defined over a finite separable extension  $L'$  of  $L$  [1: 8.11] and form a countable set. For  $\lambda \in X^*(T)$ ,  $\lambda \neq 1$ , let  $T_\lambda = \ker \lambda$ , and  $V_\lambda$  the Zariski-closure of  ${}^G T_\lambda$ . The  $V_\lambda$  and  $S$  form a countable set  $\mathcal{V}$  of proper algebraic subsets of  $G$  which are all defined over  $\bar{L}$ .

Our assertion is now a special case of the Theorem.

3. We can now prove the Corollary in the introduction. Let  $\Omega$  be an algebraically closed extension of  $K$ . Since  $G(K)/H(K)$  may be identified to an orbit of  $G(K)$  in  $G(\Omega)/H(\Omega)$  it suffices to show:

COROLLARY 2. *Assume  $K$  to be algebraically closed. Then every  $\gamma \in \Gamma - \{1\}$ , operating by left translations on  $G(K)/H(K)$ , has exactly  $\chi(G, H)$  fixed points.*

For  $\gamma \in \Gamma - \{1\}$ , let  $F_\gamma$  be the fixed point set of  $\gamma$  in  $G(K)/H(K)$ , and let  $T_\gamma$  be the maximal torus in which the cyclic group generated by  $\gamma$  is dense. Clearly,  $F_\gamma$  is also the set of fixed points of  $T_\gamma(K)$ . Thus, if  $F_\gamma$  is non-empty, then  $T_\gamma$  is conjugate to a subgroup of  $H$ , and  $H$  has maximal rank. Assume this is the case and let  $T_0$  be a maximal  $K$ -torus of  $H$ . Since the maximal tori of  $H$  (or  $G$ ) are conjugate, it is elementary that  $F_\gamma$  may be identified with  $\text{Tr}(T_0, T_\gamma)/N_H(T_0)$ . But, if  $x \in \text{Tr}(T_0, T_\gamma)$ , then  $\text{Tr}(T_0, T_\gamma) = x \cdot N_G(T_0)$ , whence the Corollary.

4. We now generalize slightly the Corollary in case  $H$  contains a maximal torus of  $G$ , dropping again the assumption that  $K$  is algebraically closed. Assume instead

(\*) *The maximal  $K$ -tori of  $H$  are conjugate under  $H(K)$ .*

If  $T_0$  is a maximal  $K$ -torus of  $H$ , we then set

$$\chi(G(K), H(K)) = [N_{G(K)}(T_0) : N_{H(K)}(T_0)].$$

If  $K$  is algebraically closed, then (\*) is fulfilled and  $\chi(G(K), H(K))$  is our previous  $\chi(G, H)$ . We again set  $\chi(G(K), H(K)) = 0$  if  $H$  does not contain any maximal torus of  $G$ .

**COROLLARY 3.** *Let  $\Gamma$  be as in Theorem 2. Let  $H$  be a closed  $K$ -subgroup of maximal rank and assume (\*) to be satisfied. Then  $\gamma \in \Gamma - \{1\}$  acts freely if  $T_\gamma$  is not conjugate under  $G(K)$  to  $T_0$  and has  $\chi(G(K), H(K))$  fixed points otherwise.*

The argument is the same as before:  $F_\gamma$  is also the set of fixed points of  $T_\gamma$ . The latter is defined over  $K$ . If  $F_\gamma \neq \emptyset$ , then there exists  $x \in G(K)$  such that  ${}^x T_\gamma \in H$ , hence by (\*),

$$\mathrm{Tr}_{G(K)}(T_0, T_\gamma) \neq \emptyset,$$

and we have, as above, bijections

$$F_\gamma = \mathrm{Tr}_{G(K)}(T_0, T_\gamma)/N_{H(K)}(T_0) = N_{G(K)}(T_0)/N_{H(K)}(T_0).$$

5. (i) If  $K = \mathbf{R}, \mathbf{C}$  or also is a non-archimedean local field with finite residue field, then  $G(K)$ , endowed with the topology stemming from  $K$ , is a Lie group over  $K$ , and in particular is a locally compact topological group. In that case, we can use in Theorem 2 a category argument instead of Lemma 2: the  $f_w^{-1}(Z)$ , being proper algebraic subsets, have no interior point, the intersection of their complement is then dense by Baire's theorem, whence the last assertion of Theorem 2 with "Zariski-dense" replaced by "dense in the  $K$ -topology".

(ii) In [4] it is asked whether the hyperbolic  $n$ -space admits a non-commutative free group of isometries which acts freely. More generally, one has the

**PROPOSITION.** *Let  $S$  be a connected semi-simple non-compact Lie group with finite center,  $U$  a maximal compact subgroup of  $L$  and  $X = L/U$  the symmetric space of non-compact type of  $S$ . Then  $S$  contains a non-commutative free subgroup which acts freely on  $X$ .*

If  $\mathrm{rk} S \neq \mathrm{rk} U$ , this could be deduced from Corollary 2. However, the existence of one such subgroup can be proved much more directly in all cases: if  $S = \mathbf{SL}_2(\mathbf{R})$  or  $\mathbf{PSL}_2(\mathbf{R})$ , then we may take for  $\Gamma$  a free subgroup of finite index in  $\mathbf{SL}_2(\mathbf{Z})$  or  $\mathbf{SL}_2(\mathbf{Z})/\{\pm 1\}$ . If  $S$  is of dimension  $> 3$ , then it contains a copy of  $\mathbf{SL}_2(\mathbf{R})$  or of  $\mathbf{PSL}_2(\mathbf{R})$ , and therefore a discrete non-commutative free subgroup  $\Gamma$ . No element  $\gamma \in \Gamma - \{1\}$  is contained in a compact subgroup of  $S$ , hence  $\Gamma$  acts freely on  $X$ .

A similar argument would be valid over a non-archimedean local field  $K$  for the Bruhat-Tits buildings attached to semi-simple  $K$ -groups.