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The condition on  $H$  and the second alternative hold either if  $K$  is algebraically closed or if  $K = \mathbf{R}$  and  $G(K)$  is compact. In that last case,  $\chi(G(K), H(K)) = \chi(G(K)/H(K))$  by [10], and Theorem A follows.

I wish to thank D. Sullivan for having sent me a preprint of [8], which was the starting point of the present paper, and D. Kazhdan and G. Prasad for having pointed out two errors in a previous proof of Theorem B for  $\mathbf{SL}_n$ .

*Notation and conventions.* In the sequel,  $G$  is a connected semi-simple algebraic group over some groundfield, and  $p$  the characteristic of the groundfield. For unexplained notation and notions on linear algebraic groups, we refer to [1]. In particular, in such a group, the word "torus" is meant as in [1], i.e., refers to a connected linear algebraic group which is isomorphic to a product of  $\mathbf{GL}_1$ 's. In a compact group however it means a topological torus (product of circle groups).

If  $H$  is a group, and  $A, B$  are subsets of  $H$ , then

$${}^B A = \{bab^{-1} \mid a \in A, b \in B\}, \quad N_H(A) = \{h \in H \mid hAh^{-1} = A\},$$

$$\mathrm{Tr}_H(A, B) = \{h \in H \mid h.A.h^{-1} = B\}.$$

If  $\Gamma$  acts on a space  $X$ , the *isotropy group* of  $\Gamma$  at  $x$  is

$$\Gamma_x = \{\gamma \in \Gamma \mid \gamma \cdot x = x\}.$$

We recall that a morphism  $f : X \rightarrow Y$  of irreducible algebraic varieties is *dominant* if its image is not contained in any proper algebraic subvariety. If so, then  $\mathrm{Im} f$  contains a Zariski-dense open subset of  $Y$  [1: AG 10.2]. If the groundfield has characteristic zero, then, since  $f$  is separable, the differential of  $f$  has maximal rank on some non-empty Zariski open subset of  $X$  [1: AG, 17.3].

## §1. PROOF OF THEOREM B

Let  $m$  be an integer  $\geq 2$ . Let  $w = w(X_1, \dots, X_m)$  be a non-trivial element in the free group  $F(X_1, \dots, X_m)$  on  $m$  letters  $X_i$ , i.e., a non-trivial reduced word in the  $X_i$ 's, with non-zero integral exponents [3: I.81, Prop. 7]. Then given a group  $H$ , the word  $w$  defines a map  $f_w : H^m \rightarrow H$  by the rule

$$(1) \quad f_w(\{h_1, \dots, h_m\}) = w(h_1, \dots, h_m), \quad (h_i \in H; 1 \leq i \leq m).$$

If  $H$  is an algebraic group, then  $f_w$  is a morphism of algebraic varieties which is defined over any field of definition for  $H$ . In the case where  $H = G$  we want to prove

THEOREM 1. *The map  $f_w : G^m \rightarrow G$  is dominant.*

This is a geometric statement. To prove it, we shall identify  $G$  with  $G(\Omega)$ , where  $\Omega$  is some universal field. We have then to prove that  $f_w(G(\Omega)^m)$  is Zariski-dense in  $G(\Omega)$ .

The Zariski closure  $Z$  of  $\text{Im } f_w$  is irreducible (since  $G^m$  is) and is invariant under conjugation, since  $\text{Im } f_w$  is obviously so. Since the semi-simple elements of  $G$  are Zariski-dense, and all conjugate to elements in some fixed maximal torus  $T$ , it suffices to show that  $Z \supset T$ .

a) We first consider the case where  $G = \text{SL}_n (n \geq 2)$ . Let us prove that  $G(\Omega)$  contains a Zariski-dense subgroup  $H$ , no element of which, except for the identity, has an eigenvalue equal to one. This statement and its proof were directly suggested by [8].

One can find an infinite field  $L$  of the same characteristic as  $\Omega$  over which there exists a central division algebra  $D$  of degree  $n^2$ . We may for example take for  $L$  a local field (see e.g. XIII, §3, Remarque p. 202 in [14]). We may assume  $L \subset \Omega$ . Let  $\mathcal{D}^1$  be the algebraic group over  $L$  whose points in a commutative  $L$ -algebra  $M$  are the elements of reduced norm one in  $D \otimes_L M$ . Then  $\mathcal{D}^1$  is an anisotropic  $L$ -form of  $\text{SL}_n$ . Of course,  $D$  splits over  $\Omega$  and the isomorphism  $D \otimes_L \Omega = \mathbf{M}_n(\Omega)$  yields an isomorphism of  $\mathcal{D}^1(\Omega)$  onto  $G(\Omega)$ . We let  $H$  be the image of  $D^1 = \mathcal{D}^1(L)$  under such an isomorphism. The group  $H$  is Zariski-dense since  $L$  is infinite. The fact that any  $h \in H - \{1\}$  has no eigenvalue equal to one is then proved as in [8]: the element  $h - 1$  is a non-zero element of  $D$ , hence is invertible, hence has no eigenvalue zero and therefore  $h$  has no eigenvalue one. This proves our assertion. Let  $p_0$  be the characteristic exponent of  $\Omega$  ( $p_0 = 1$  if  $\text{char } \Omega = 0$  and  $p_0 = \text{char } \Omega$  otherwise). If  $p_0 = 1$ , then  $H$  consists of semi-simple elements; if not, then  $h^q (q = p_0^{n-1})$  is semi-simple for any  $h \in G$ . Let  $f_w^q : G^m \rightarrow G$  be defined by  $f_w^q(g) = f_w(g)^q$ . Then  $f_w^q(H)$  consists of semi-simple elements. Let  $Z_q$  be the Zariski closure of  $\text{Im } f_w^q$ . Since  $x \mapsto x^q$  is dominant, we have shown :

(\*) *Let  $V$  be the set of semi-simple elements in  $G(\Omega)$  which have no eigenvalue equal to one. Then  $\{1\} \cup (V \cap \text{Im } f_w^q)$  is Zariski-dense in  $Z_q$ .*

We now prove the theorem for  $\text{SL}_n (n \geq 2)$  by induction on  $n$ . It suffices to show that  $f_w^q$  is dominant and, for this, that  $Z_q \supset T$ . Let  $n = 2$ . The group  $\text{SL}_2$  has dimension three and the conjugacy classes of non-central elements have dimension two. If  $Z_q \neq G$ , then  $\dim Z_q \leq 2$  and  $Z_q$  is contained in the union of the set  $U$  of unipotent elements of  $G$  and of finitely many conjugacy classes of semi-simple elements  $\neq 1$ . Those are closed, disjoint from  $U$ . Since  $Z_q$  is irreducible and contains 1, it should then be contained in  $U$ . On the other hand,

$Z_q \neq \{1\}$  since  $G$  contains non-commutative free subgroups, as follows from [17] (see also Remark 1 below). We then get

$$\{1\} \subsetneq Z_q \subset U,$$

but this contradicts (\*), whence the Theorem for  $\mathbf{SL}_2$ .

Assume now  $n > 2$  and our assertion proved up to  $n - 1$ . This implies in particular that  $Z_q$  contains all subgroups of  $G$  isomorphic to  $\mathbf{SL}_{n-1}$ , hence that  $Z_q \cap T$  contains the subtori of  $T$  of codimension one consisting of the elements of  $T$  which have at least one eigenvalue equal to one. Call  $Y$  their union. Assume that  $Z_q \cap T \neq T$ . Then we may write  $Z_q \cap T = Y \cup Y'$ , where  $Y'$  is a proper algebraic subset of  $T$  not containing any irreducible component of  $Y$ . Let  $Q$  be the Zariski-closure of the set  ${}^G Y'$  of conjugates of elements of  $Y'$ . We claim that  $Y \not\subset Q$ . In fact, the subsets  $Y$  and  $Y'$  are stable under the Weyl group  $W = N(T)/T$  (which may be identified with the group of permutations of the basic vectors of  $\Omega^n$ ). Let  $J \subset \Omega[T/W]$  be the ideal of  $Y'$ . The algebra  $\Omega[T/W]$  is isomorphic, under the restriction mapping, to the algebra  $S$  of regular class functions on  $G$  [16]. Let  $J'$  be the ideal of  $S$  corresponding to  $J$  under this isomorphism and  $R$  the variety of zeroes of  $J'$ . We have then  $Q \subset R$ , but  $Y \not\subset R$ , whence  $Y \not\subset Q$ .

The difference  $Y' - (Y \cap Y')$  contains a conjugate of every semi-simple element of  $Z_q$  not having any eigenvalue equal to one. Therefore (\*) implies that  $Z_q = \{1\} \cup Q$ . But this contradicts the fact that  $Y \not\subset Q$ . Therefore  $T \subset Z_q$  and the theorem is proved for  $\mathbf{SL}_n$ .

b) In the general case we use induction on  $\dim G$ . If  $\mu : G' \rightarrow G$  is an isogeny, then the theorem for  $G'$  implies it for  $G$ , hence we may assume  $G$  to be simply connected. It is then a direct product of almost simple groups, whence also a reduction to the case where  $G$  is almost simple. By a), it suffices to consider the case where  $G$  is not isomorphic to  $\mathbf{SL}_n$  for any  $n$ . But then it contains a proper connected semi-simple subgroup  $H$  of maximal rank (see lemma below). By induction  $Z$  contains a maximal torus of  $H$ , hence one of  $G$ , and therefore  $T$ .

We have just used the following lemma:

LEMMA 1. *Assume  $G$  to be almost simple, and not isogeneous to  $\mathbf{SL}_n$  for any  $n$ . Then  $G$  contains a proper connected semi-simple subgroup of maximal rank.*

For convenience, we may assume  $G$  to be isomorphic to its adjoint group. Let  $\Phi = \Phi(G, T)$  be the root system of  $G$  with respect to  $T$  and  $\Delta = \{\alpha_1, \dots, \alpha_l\}$  a

basis of  $\Phi$ . Since  $G$  is adjoint,  $\Delta$  is also a basis of the group  $X^*(T)$  of rational characters of  $T$ . Let  $d$  be the dominant root and write

$$d = \sum_{i=1}^{i=l} d_i \alpha_i.$$

The  $d_i$ 's are strictly positive integers. By assumption,  $\Phi$  is not of type  $A_m$  for any  $m$ . Therefore, by the classification of root systems, one of the  $d_i$ 's is prime (see e.g. [4]). Say  $d_1 = q$ , with  $q$  prime. Let  $\Psi$  be the set of elements in  $\Phi$  which, when expressed as linear combination of simple roots, have either 0 or  $\pm q$  as coefficient of  $\alpha_1$ . This is a closed set of roots. In fact, it is a root system with basis  $\alpha_2, \dots, \alpha_l$  and  $-d$  [2]. We claim that there exists a closed connected subgroup  $H$  of  $G$  containing  $T$  with root system  $\Psi$ .

Let first  $q \neq \text{char. } K$ . Then there is an element  $t \in T$ ,  $t \neq 1$ , such that

$$d(t) = \alpha_i(t) = 1, \quad (i = 2, \dots, l).$$

It has order  $q$ , and  $\Psi$  is the set of roots which are equal to one on  $t$ . Then the identity component of the centralizer of  $t$  satisfies our condition.

Let now  $q = \text{char. } \Omega$ . Let  $\mathfrak{t}$  be the Lie algebra of  $T$  and  $\mathfrak{u}$  be the subspace of  $\mathfrak{t}$  which annihilates the differentials  $d\alpha_i$  of the roots  $\alpha_i$  ( $i = 2, \dots, l$ ). It is one dimensional and does not annihilate  $d\alpha_1$  (since, as recalled above,  $\Delta$  is a basis of  $X^*(T)$ , hence the  $d\alpha_i$  ( $1 \leq i \leq l$ ) form a basis of the dual space to  $\mathfrak{t}$ ). Of course, the differential of any  $\lambda \in X^*(T)$  which is divisible by  $q$  in  $X^*(T)$  is identically zero on  $\mathfrak{t}$ . It follows then that

$$\Psi = \{\alpha \in \Phi \mid d\alpha(\mathfrak{u}) = 0\}.$$

Let  $\mathfrak{g}$  be the Lie algebra of  $G$  and

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \text{Ad } t(x) = \alpha(t) \cdot x (t \in T)\}, \quad (\alpha \in \Phi),$$

be the (1-dimensional) eigenspace of  $T$  corresponding to  $\alpha$  [1, §14]. The previous relation implies that

$$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{u}) = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha.$$

By [1: §14] the Lie algebra of the centralizer

$$Z_G(\mathfrak{u}) = \{g \in G \mid \text{Ad } g(x) = x, (x \in \mathfrak{u})\},$$

of  $\mathfrak{u}$  in  $G$  is equal to  $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{u})$ ; therefore  $Z_G(\mathfrak{u})$  is a semi-simple subgroup satisfying our conditions.

*Remarks.* 1) We have used [17] only for  $\text{SL}_2(\Omega)$ , but it is possible to bypass [17] in this case and make our proof, and the whole paper, independent of [17].

We need only to prove that  $\mathbf{SL}_2(\Omega)$  contains a non-commutative free subgroup  $F$ . If  $\Omega$  has characteristic zero, we may take any torsion-free subgroup of  $\mathbf{SL}_2(\mathbf{Z})$ . Let now  $p = \text{char } \Omega$  be  $> 0$ . Then, by the arithmetic method, using division quaternion algebras over global fields, we can construct a discrete cocompact subgroup of  $\mathbf{SL}_2(L)$ , where  $L$  is a local field of characteristic  $p$  (cf. A. Borel-G. Harder, *Crelle J.* 298 (1978), 53-74). The latter has a torsion-free subgroup  $F$  of finite index (H. Garland, *Annals of Math.* 97 (1973), 375-423) which is then free, since it acts freely on a tree, namely the Bruhat-Tits building of  $\mathbf{SL}_2(L)$ .

2) For any non-zero  $n \in \mathbf{Z}$ , the power map  $g \mapsto g^n$  is dominant (because it is surjective on any maximal torus [1 : 8.9]), hence Theorem 1 is obvious if the sum of the exponents of one letter in the word  $w$  is not zero. (See [11] for a similar remark in the context of compact groups.)

3) If  $U$  and  $V$  are non-empty open subsets in a connected algebraic group  $H$ , then  $H = U \cdot V$  [1 : 1.3]. It follows then from Theorem 1 that if  $w, w'$  are two words in two letters, say, then the map  $G^4 \rightarrow G$  defined by

$$f(g_1, g_2, g_3, g_4) = w(g_1, g_2) \cdot w'(g_3, g_4),$$

is surjective. For instance, every element of  $G(\Omega)$  is the product of two commutators. However, the map  $f_w$  itself is not always surjective; for instance  $x \mapsto x^2$  is not surjective in  $\mathbf{SL}_2(\mathbf{C})$ , as pointed out in [11].

4) If  $K = \mathbf{C}$ , then Theorem 1 implies that  $\text{Im } f_w$  contains a dense open set in the ordinary topology. If  $G$  is defined over  $\mathbf{R}$ , then Theorem 1 also shows that  $f_w(G(\mathbf{R}))$  contains a non-empty subset of  $G(\mathbf{R})$  which is open in the ordinary topology. However it may not be dense. For instance, it is pointed out in [11] that for  $\mathbf{SU}_2$ , the image of the map defined by  $[x^2, yxy^{-1}]$  omits a neighborhood of  $-1$ ; however this map is surjective in  $\mathbf{SO}_3$ .

It seems that little is known about the image of  $f_w$ , even over  $\mathbf{R}$  or  $\mathbf{C}$ . A general fact however is that the commutator map is surjective in any compact connected semi-simple Lie group [9].

## §2. FREE SUBGROUPS WITH STRONGLY REGULAR ELEMENTS

1. In the sequel,  $K$  is a field of infinite transcendence degree over its prime field. We shall need the following lemma:

LEMMA 2. *Let  $X$  be an irreducible unirational  $K$ -variety. Let  $L$  be a finitely generated subfield of  $K$  containing a field of definition of  $X$ , and  $V_i (i \in \mathbf{N})$  a sequence of proper irreducible algebraic subsets of  $X$  defined over an*