This shows that D has involutions of the second kind if and only if the class of D_1 in $\mathrm{Br}(k)$ is trivial. This class is the required norm $N_{K/k}(\mathrm{cl}(D))$. In the localization spirit, this can be deduced from the fact that the homothety by $\lambda \in K'^*$ of V' induces on W' the homothety by $N_{K'/k'}(\lambda) \in k'^*$.

Appendix C

For $n \geqslant 3$, examples can be obtained as follows: take $k' = \mathbf{Q}[\zeta]$, with $\zeta = \exp(2\pi i/n)$, and $k = k' \cap \mathbf{R}$. Fix $a, b \in k^*$ and let D be the k'-algebra generated by X, Y, subject to

$$X^n = a\,, \qquad Y^n = b$$
$$XY = \zeta YX\,.$$

It admits the anti-involution $*$, inducing complex conjugation on k', defined by $\zeta^* = \zeta^{-1}$, $X^* = X$, $Y^* = Y$. The algebra D is of the type we require, provided it is a division algebra. This happens already with $a, b \in \mathbf{Z}$: take for a a prime congruent to 1 mod n, and for b an integer whose residue mod a has in the cyclic group of order n $(\mathbf{Z}/(a))^*/(\mathbf{Z}/(a))^{*n}$ an image of exact order n. For instance $n = 3$, $a = 7$, $b = 2$. For $n = 2$, one proceeds similarly with $k' = \mathbf{Q}[i]$, $\zeta = -1$, a congruent to 1 mod 4 and b not a square mod a. For instance, $a = 5$, and $b = 2$. In each case, the assumption on a ensures that k' embed in the a-adic completion $\mathbf{Q}_a$ of $\mathbf{Q}$, and the fact that D is a division algebra can be seen locally at a : $D \otimes_{k'} \mathbf{Q}_a$ is a division algebra with center $\mathbf{Q}_a$.