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of  $U(n)$  is not trivial, and  $U(n)$  is not solvable. The group  $\Gamma$  is dense in  $U(n)$ : skew adjoints elements of  $D$  are dense in the skew adjoint matrices in  $M(n, \mathbf{C})$ , and the Cayley transform  $t \mapsto \frac{t - 1}{t + 1}$  is an homeomorphism from the space of skew-adjoint matrices in  $M(n, \mathbf{C})$  to an open dense subset of  $U(n)$ , carrying skew adjoint elements of  $D$  into  $\Gamma$ . From this density, it results that, if  $n > 1$ , the linear group  $\Gamma$  is not solvable. By [Tits], it contains a non abelian free subgroup.

It remains to construct pairs  $(D, *)$ . A division algebra  $D$  with center  $k'$  admits an anti-involution  $*$  inducing on  $k'$  the non trivial element  $\text{Gal}(k'/k)$ , if and only if its class  $\text{cl}(D)$  in the Brauer group  $\text{Br}(k')$  of  $k'$  has a trivial image by the norm map  $N_{k'/k} : \text{Br}(k') \rightarrow \text{Br}(k)$ —see Appendix B. Class field theory provides an explicit computation of  $\text{Br}(k)$ , and of  $N_{k'/k}$ , and tells which elements of  $\text{Br}(k')$  come from division algebras. From the explicit description it provides, existence of such  $D$  follows. A direct construction is given in Appendix C. When we choose an isomorphism of  $D \otimes_k \mathbf{R}$  with  $M(n, \mathbf{C})$ , the involution  $*$  becomes adjunction with respect to some hermitian form  $\phi$  on  $\mathbf{C}^n$ , not necessarily positive definite:  $\phi(ax, y) = \phi(x, a^*y)$ . If  $h$  is self adjoint in  $D$ ,  $\text{int}(h^{-1}) \circ *$  is adjunction, with respect to the form  $\phi_h(x, y) = \phi(hx, y)$ . For suitable  $h$ ,  $\phi_h$  is positive definite and  $(D, \text{int}(h^{-1}) \circ *)$  is of the type sought.

### APPENDIX A

Consider  $\phi : S' \cup S'' \rightarrow S - E$  as in the introduction, with  $S'$  and  $S''$  two copies of the sphere  $S$ , and  $\psi : S \rightarrow S'$  the obvious bijection. Consider as in the Schröder-Bernstein theorem the set  $S_e$  of points  $p$  in  $S$  with an even number of ancestors, namely for which there exists an integer  $n \geq 0$  with  $p \in \text{Im}(\phi \circ \psi)^n$  and  $p \notin \text{Im}(\psi \circ (\phi \circ \psi)^n)$ . Consider also the set  $S_0$  of those  $p$  in  $S$  for which there exists  $n \geq 0$  with  $p \in \text{Im}(\psi \circ (\phi \circ \psi)^n)$  and  $p \notin \text{Im}(\phi \circ \psi)^{n+1}$ , and finally the set  $S_\infty$  of those  $p$  such that  $p \in \text{Im}(\phi \circ \psi)^n$  for any  $n \geq 0$ . Consider similarly

$$S' \cup S'' = (S' \cup S'')_e \cup (S' \cup S'')_0 \cup (S' \cup S'')_\infty .$$

Then  $\psi$  induces a bijection from  $S_e \cup S_\infty$  onto  $(S' \cup S'')_0 \cup (S' \cup S'')_\infty$  and  $\phi^{-1}$  from  $S_0$  onto  $(S' \cup S'')_e$ . Combining these two we have a bijection  $\chi : S \rightarrow S' \cup S''$  and a partition of  $S$  into finitely many pieces, the restriction of  $\chi$  to each of these being a rotation.