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Anhang: Appendix A

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of $U(n)$ is not trivial, and $U(n)$ is not solvable. The group Γ is dense in $U(n)$: skew adjoints elements of D are dense in the skew adjoint matrices in $M(n, \mathbf{C})$, and the Cayley transform $t \mapsto \frac{t-1}{t+1}$ is an homeomorphism from the space of skew-adjoint matrices in $M(n, \mathbf{C})$ to an open dense subset of $U(n)$, carrying skew adjoint elements of D into Γ. From this density, it results that, if $n > 1$, the linear group Γ is not solvable. By [Tits], it contains a non abelian free subgroup.

It remains to construct pairs $(D, *)$. A division algebra D with center k' admits an anti-involution $*$ inducing on k' the non trivial element $\mathrm{Gal}(k'/k)$, if and only if its class $\mathrm{cl}(D)$ in the Brauer group $\mathrm{Br}(k')$ of k' has a trivial image by the norm map $N_{k'/k}: \mathrm{Br}(k') \to \mathrm{Br}(k)$—see Appendix B. Class field theory provides an explicit computation of $\mathrm{Br}(k)$, and of $N_{k'/k}$, and tells which elements of $\mathrm{Br}(k')$ come from division algebras. From the explicit description it provides, existence of such D follows. A direct construction is given in Appendix C. When we choose an isomorphism of $D \otimes_k \mathbf{R}$ with $M(n, \mathbf{C})$, the involution $*$ becomes adjunction with respect to some hermitian form ϕ on $\mathbf{C}^n$, not necessarily positive definite: $\phi(ax, y) = \phi(x, a^*y)$. If h is self adjoint in D, $\mathrm{int}(h^{-1}) \circ *$ is adjunction, with respect to the form $\phi_h(x, y) = \phi(hx, y)$. For suitable h, ϕ_h is positive definite and $(D, \mathrm{int}(h^{-1}) \circ *)$ is of the type sought.

Appendix A

Consider $\phi : S' \cup S'' \to S - E$ as in the introduction, with S' and S'' two copies of the sphere S, and $\psi : S \to S'$ the obvious bijection. Consider as in the Schröder-Bernstein theorem the set S_e of points p in S with an even number of ancestors, namely for which there exists an integer $n \geqq 0$ with $p \in \mathrm{Im}(\phi\circ\psi)^n$ and $p \notin \mathrm{Im}(\psi \circ (\phi\circ\psi)^n)$. Consider also the set S_0 of those p in S for which there exists $n \geqq 0$ with $p \in \mathrm{Im}(\psi \circ (\phi\circ\psi)^n)$ and $p \notin \mathrm{Im}(\phi\circ\psi)^{n+1}$, and finally the set S_∞ of those p such that $p \in \mathrm{Im}(\phi\circ\psi)^n$ for any $n \geqq 0$. Consider similarly

$$S' \cup S'' = (S'\cup S'')_e \cup (S'\cup S'')_0 \cup (S'\cup S'')_\infty .$$

Then ψ induces a bijection from $S_e \cup S_\infty$ onto $(S'\cup S'')_0 \cup (S'\cup S'')_\infty$ and ϕ^{-1} from S_0 onto $(S'\cup S'')_e$. Combining these two we have a bijection $\chi : S \to S' \cup S''$ and a partition of S into finitely many pieces, the restriction of χ to each of these being a rotation.