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# DIVISION ALGEBRAS AND THE HAUSDORFF-BANACH-TARSKI PARADOX

by Pierre Deligne and Dennis Sullivan

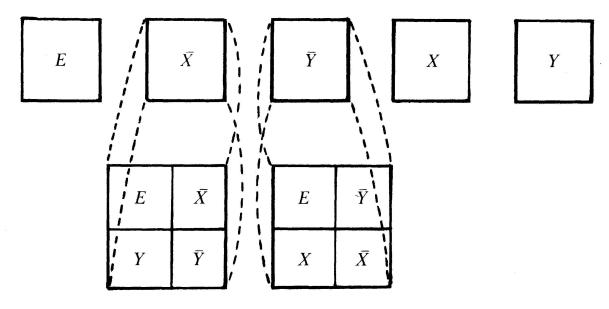
In this note we observe that a question raised by Dekker (1956) about rotations inspired by the Hausdorff-Banach-Tarski paradox can be answered using algebraic number theory. For motivation, we recall a form of the paradox.

Partition the free group in two generators F into the five sets E, X,  $\bar{X}$ , Y,  $\bar{Y}$  consisting respectively of the identity and of the elements which, when written in reduced form, begin with x,  $x^{-1}$ , y or  $y^{-1}$ . If F acts freely on a sphere S, by rigid rotations, and if, using the axiom of choice, we choose a transversal T, i.e. a set with exactly one point in each orbit, the five subsets T, XT,  $\bar{X}T$ , YT,  $\bar{Y}T$  form a partition of S. For economy of notation, we will write again E, X,  $\bar{X}$ ,  $\bar{Y}$ ,  $\bar{Y}$  for T, XT,  $\bar{X}T$ , YT,  $\bar{Y}T$ . The rotation by x moves  $\bar{X}$  onto

$$S - X = E \cup \bar{X} \cup Y \cup \bar{Y}.$$

Similarly, y moves  $\bar{Y}$  onto

$$S - Y = E \cup X \cup \bar{X} \cup \bar{Y}.$$



Thus we can reassemble from these 11 (actually 5) pieces 2 congruent spheres plus one congruent copy of the set E. This is a form of the Hausdorff-Banach-Tarski paradox which comes quickly from a free action of a free group

on two generators (see Appendix A for the more precise form). Thus we have the question of Dekker (communicated by Jan Mycielski): do such actions really exist? The sphere must be odd-dimensional for topological reasons: a fixed point free map  $f: S^d \to S^d$  must have vanishing Lefschetz number

$$L(f) = \sum (-1)^i \operatorname{Trace}(f; H_i(S^d)) = 1 + (-1)^d \operatorname{deg}(f).$$

For a free action of a free group, the square of a non-trivial element will act by a map of degree +1 and this forces d odd. The dimension must be >1. These are the only conditions.

THEOREM. For  $n \ge 2$ , there is a free non-abelian group of rigid rotations acting freely on the odd dimensional sphere  $S^{2n-1}$ .

Remark. The corresponding orthogonal matrices can be chosen to have algebraic entries, and the group of matrices corresponds to a subgroup of the non-zero elements in a division algebra over a number field.

*Remark.* The theorem was proved by Dekker for *n* even [D].

*Remark.* Let  $u: O(p) \times O(q) \rightarrow O(p+q)$  be the natural embedding:

$$u(A, B) = \max \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

If  $A \in O(p)$  and  $B \in O(q)$  don't have any nonzero fixed vector, neither has u(A, B). If  $\sigma_1 : F \to O(p)$  and  $\sigma_2 : F \to O(q)$  define free actions on  $S^{p-1}$  and  $S^{q-1}$ ,  $u(\sigma_1, \sigma_2)$  hence defines a free action of F on  $S^{p+q-1}$ . Using this remark, one could reduce the theorem to the two particular cases n = 2 and n = 3.

*Proof.* Let  $k \subset \mathbf{R}$  be a real algebraic number field, and  $k' \subset \mathbf{C}$  be a quadratic extension of k. We assume that  $k' \not\in \mathbf{R}$ , i.e. that  $k' \otimes_k \mathbf{R} = \mathbf{C}$ . Let D be a division algebra of dimension  $n^2$  over its center k', equipped with an anti-involution \* inducing on k' the complex conjugation. The  $\mathbf{R}$ -algebra  $D \otimes_k \mathbf{R}$  is a simple algebra over its center  $k' \otimes_k \mathbf{R} = \mathbf{C}$ , hence isomorphic to  $M(n, \mathbf{C})$ . We assume that, for a suitable isomorphism between  $D \otimes_k \mathbf{R}$  and  $M(n, \mathbf{C})$ , \* becomes transpose conjugate.

In term of an isomorphism as above, the elements u of D satisfying  $uu^* = 1$  become unitary matrices. They operate on the unit sphere in  $\mathbb{C}^n$ . Furthermore, if  $u \neq 1$ , u - 1 is invertible in D so that the corresponding matrix does not have 1 as an eigenvalue. It hence acts without fixed point on the sphere.

The group  $\Gamma := \{u \in D \mid uu^* = 1\}$  is the group of k-rational points of a k-form of the real algebraic group U(n). For n > 1, the perfect subgroup SU(n)

of U(n) is not trivial, and U(n) is not solvable. The group  $\Gamma$  is dense in U(n): skew adjoints elements of D are dense in the skew adjoint matrices in  $M(n, \mathbb{C})$ , and the Cayley transform  $t \mapsto \frac{t-1}{t+1}$  is an homeomorphism from the space of skew-adjoint matrices in  $M(n, \mathbb{C})$  to an open dense subset of U(n), carrying skew adjoint elements of D into  $\Gamma$ . From this density, it results that, if n > 1, the linear group  $\Gamma$  is not solvable. By [Tits], it contains a non abelian free subgroup.

It remains to construct pairs (D, \*). A division algebra D with center k' admits an anti-involution \* inducing on k' the non trivial element  $\operatorname{Gal}(k'/k)$ , if and only if its class  $\operatorname{cl}(D)$  in the Brauer group  $\operatorname{Br}(k')$  of k' has a trivial image by the norm map  $N_{k'/k}:\operatorname{Br}(k')\to\operatorname{Br}(k)$ —see Appendix B. Class field theory provides an explicit computation of  $\operatorname{Br}(k)$ , and of  $N_{k'/k}$ , and tells which elements of  $\operatorname{Br}(k')$  come from division algebras. From the explicit description it provides, existence of such D follows. A direct construction is given in Appendix C. When we choose an isomorphism of  $D\otimes_k \mathbf{R}$  with  $M(n, \mathbf{C})$ , the involution \* becomes adjunction with respect to some hermitian form  $\Phi$  on  $\mathbf{C}^n$ , not necessarily positive definite:  $\Phi(ax, y) = \Phi(x, a^*y)$ . If h is self adjoint in D,  $\operatorname{int}(h^{-1}) \circ *$  is adjunction, with respect to the form  $\Phi(x, y) = \Phi(hx, y)$ . For suitable h,  $\Phi(x) = \Phi(x)$  is positive definite and  $\Phi(x) = \Phi(x)$  is of the type sought.

### APPENDIX A

Consider  $\phi: S' \cup S'' \to S - E$  as in the introduction, with S' and S'' two copies of the sphere S, and  $\psi: S \to S'$  the obvious bijection. Consider as in the Schröder-Bernstein theorem the set  $S_e$  of points p in S with an even number of ancestors, namely for which there exists an integer  $n \ge 0$  with  $p \in \operatorname{Im}(\phi \circ \psi)^n$  and  $p \notin \operatorname{Im}(\psi \circ (\phi \circ \psi)^n)$ . Consider also the set  $S_0$  of those p in S for which there exists  $n \ge 0$  with  $p \in \operatorname{Im}(\psi \circ (\phi \circ \psi)^n)$  and  $p \notin \operatorname{Im}(\phi \circ \psi)^{n+1}$ , and finally the set  $S_\infty$  of those p such that  $p \in \operatorname{Im}(\phi \circ \psi)^n$  for any  $p \in \operatorname{Im}(\phi \circ \psi)^n$  for any  $p \in \operatorname{Im}(\phi \circ \psi)^n$  for any  $p \in \operatorname{Im}(\phi \circ \psi)^n$ 

$$S' \cup S'' = (S' \cup S'')_e \cup (S' \cup S'')_o \cup (S' \cup S'')_\infty.$$

Then  $\psi$  induces a bijection from  $S_e \cup S_\infty$  onto  $(S' \cup S'')_0 \cup (S' \cup S'')_\infty$  and  $\phi^{-1}$  from  $S_0$  onto  $(S' \cup S'')_e$ . Combining these two we have a bijection  $\chi \colon S \to S' \cup S''$  and a partition of S into finitely many pieces, the restriction of  $\chi$  to each of these being a rotation.

# APPENDIX B

Let K be a separable quadratic extension of a field k. We denote  $x \mapsto \bar{x}$  the non trivial element  $\operatorname{Gal}(K/k)$ . Let D be a simple algebra with dimension  $n^2$  over its center K. We will check the criterion of the text, for the existence of an involution of the second kind on D, i.e. of an anti-involution \* of D, inducing  $x \mapsto \bar{x}$  on K. The criterion is that  $N_{K/k} \operatorname{cl}(D) = 0$  in  $\operatorname{Br}(k)$ .

Let us localize, for the étale topology, over  $\operatorname{Spec}(k)$ . This means making large enough étale extensions of scalars  $\bigotimes_k k'$ , and keeping track of the functoriality in k'. The field K becomes the separable quadratic extension  $K' = K \bigotimes_k k'$  of k'. The algebra D becomes  $D' = D \bigotimes_k k'$ , and is of the form  $D' = \operatorname{End}_{K'}(V')$ , for V' a free module K'. The module V' is not determined uniquely by D', only up to homotheties (the corresponding projective space is uniquely determined).

For any K-module M, let  $M^-$  be the module deduced from M by the extension of scalars  $\bar{}: K \to K$ , i.e. the module, unique up to unique isomorphism, provided with an anti-linear isomorphism  $x \mapsto \bar{x}: M \cong M^-$ . Similarly for K'-modules. If  $D' = \operatorname{End}(V')$ , then  $D^{-'} = \operatorname{End}(V'^-)$ , and

$$(D \otimes_{\kappa} D^{-})' = \operatorname{End}(V' \otimes V'^{-}).$$

Let W' be the fixed subspace of the anti-linear automorphism of  $V' \otimes V'^-$  defined by  $v \otimes \bar{w} \mapsto w \otimes \bar{v}$ . It is the space of Hermitian forms on the dual of V'. One has  $W' \otimes_{k'} K' = V' \otimes V'^-$ . If  $D_1 \subset D \otimes_K D^-$  is the fixed subspace of the anti-linear automorphism of  $D \otimes_K D^-$  defined by  $x \otimes \bar{y} \mapsto y \otimes \bar{x}$ , then  $D'_1$  is the k'-form of the K'-algebra  $(D \otimes_K D^-)' = \operatorname{End}(V' \otimes V'^-)$  deduced from the k'-form W' of the K'-module  $V' \otimes V'^-$ :  $D'_1 = \operatorname{End}_{k'}(W')$ .

Involutions of the second kind on D' correspond one to one to non degenerate Hermitian forms on V', taken up to a factor (in  $k'^*$ ). Those, in turn, by the "dual form" construction, correspond to "non degenerate" elements of W'. Again, one has to take them up to a factor. The projective space P(W') over k' is determined up to unique isomorphism by D'. It is hence (this is the point of localisation) defined over  $k: P(W') = P \otimes_k k'$ , functorially in k'. The k-points of P (rather, the non degenerate points) parametrize the involutions of the second kind on D.

The functorial isomorphism  $D'_1 = \operatorname{End}_{k'}(W')$  shows that P is the form of projective space (Severi-Brauer variety) attached to  $D_1$ . It has a rational point, and is then the ordinary projective space, if and only if  $D_1$  is a matrix algebra.

This shows that D has involutions of the second kind if and only if the class of  $D_1$  in Br(k) is trivial. This class is the required norm  $N_{K/k}(cl(D))$ . In the localization spirit, this can be deduced from the fact that the homothety by  $\lambda \in K'^*$  of V' induces on W' the homothety by  $N_{K'/k'}(\lambda) \in k'^*$ .

## APPENDIX C

For  $n \ge 3$ , examples can be obtained as follows: take  $k' = \mathbf{Q}[\zeta]$ , with  $\zeta = \exp(2\pi i/n)$ , and  $k = k' \cap \mathbf{R}$ . Fix  $a, b \in k^*$  and let D be the k'-algebra generated by X, Y, subject to

$$X^n = a, Y^n = b$$
$$XY = \zeta YX.$$

It admits the anti-involution \*, inducing complex conjugation on k', defined by  $\zeta^* = \zeta^{-1}$ ,  $X^* = X$ ,  $Y^* = Y$ . The algebra D is of the type we require, provided it is a division algebra. This happens already with  $a, b \in \mathbb{Z}$ : take for a a prime congruent to 1 mod n, and for b an integer whose residue mod a has in the cyclic group of order n ( $\mathbb{Z}/(a)$ )\*/( $\mathbb{Z}/(a)$ )\*\* an image of exact order n. For instance n = 3, a = 7, b = 2. For n = 2, one proceeds similarly with  $k' = \mathbb{Q}[i]$ ,  $\zeta = -1$ , a congruent to 1 mod 4 and b not a square mod a. For instance, a = 5, and b = 2. In each case, the assumption on a ensures that k' embed in the a-adic completion  $\mathbb{Q}_a$  of  $\mathbb{Q}$ , and the fact that D is a division algebra can be seen locally at  $a : D \otimes_{k'} \mathbb{Q}_a$  is a division algebra with center  $\mathbb{Q}_a$ .

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