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## DIVISION ALGEBRAS AND THE HAUSDORFF-BANACH-TARSKI PARADOX

by Pierre DELIGNE and Dennis SULLIVAN

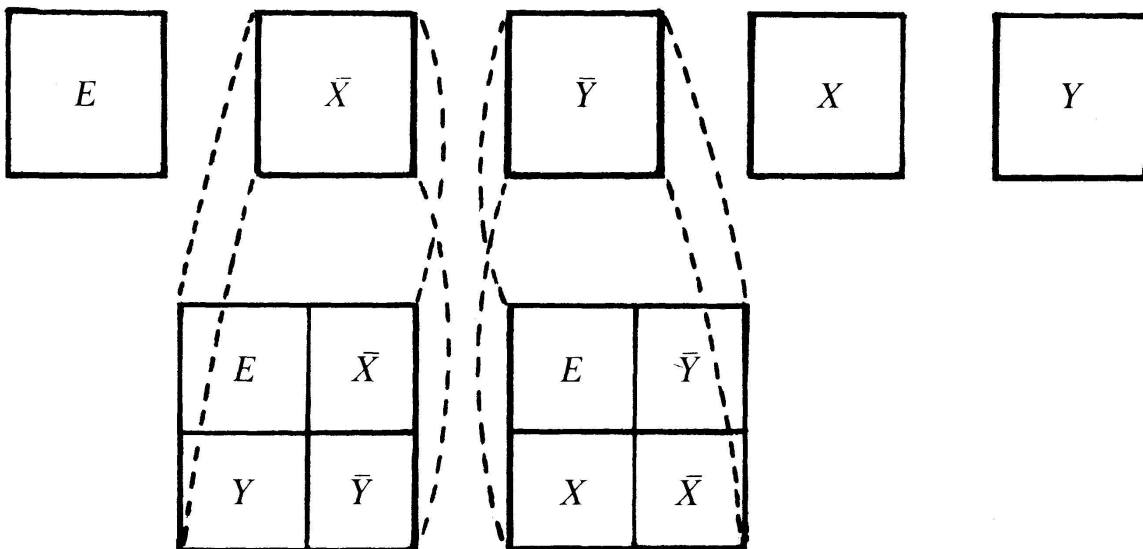
In this note we observe that a question raised by Dekker (1956) about rotations inspired by the Hausdorff-Banach-Tarski paradox can be answered using algebraic number theory. For motivation, we recall a form of the paradox.

Partition the free group in two generators  $F$  into the five sets  $E, X, \bar{X}, Y, \bar{Y}$  consisting respectively of the identity and of the elements which, when written in reduced form, begin with  $x, x^{-1}, y$  or  $y^{-1}$ . If  $F$  acts freely on a sphere  $S$ , by rigid rotations, and if, using the axiom of choice, we choose a transversal  $T$ , i.e. a set with exactly one point in each orbit, the five subsets  $T, XT, \bar{X}T, YT, \bar{Y}T$  form a partition of  $S$ . For economy of notation, we will write again  $E, X, \bar{X}, Y, \bar{Y}$  for  $T, XT, \bar{X}T, YT, \bar{Y}T$ . The rotation by  $x$  moves  $\bar{X}$  onto

$$S - X = E \cup \bar{X} \cup Y \cup \bar{Y}.$$

Similarly,  $y$  moves  $\bar{Y}$  onto

$$S - Y = E \cup X \cup \bar{X} \cup \bar{Y}.$$



Thus we can reassemble from these 11 (actually 5) pieces 2 congruent spheres plus one congruent copy of the set  $E$ . This is a form of the Hausdorff-Banach-Tarski paradox which comes quickly from a free action of a free group

on two generators (see Appendix A for the more precise form). Thus we have the question of Dekker (communicated by Jan Mycielski): do such actions really exist? The sphere must be odd-dimensional for topological reasons: a fixed point free map  $f : S^d \rightarrow S^d$  must have vanishing Lefschetz number

$$L(f) = \sum (-1)^i \text{Trace}(f ; H_i(S^d)) = 1 + (-1)^d \text{deg}(f).$$

For a free action of a free group, the square of a non-trivial element will act by a map of degree  $+1$  and this forces  $d$  odd. The dimension must be  $> 1$ . These are the only conditions.

**THEOREM.** *For  $n \geq 2$ , there is a free non-abelian group of rigid rotations acting freely on the odd dimensional sphere  $S^{2n-1}$ .*

*Remark.* The corresponding orthogonal matrices can be chosen to have algebraic entries, and the group of matrices corresponds to a subgroup of the non-zero elements in a division algebra over a number field.

*Remark.* The theorem was proved by Dekker for  $n$  even [D].

*Remark.* Let  $u : O(p) \times O(q) \rightarrow O(p+q)$  be the natural embedding:

$$u(A, B) = \text{matrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

If  $A \in O(p)$  and  $B \in O(q)$  don't have any nonzero fixed vector, neither has  $u(A, B)$ . If  $\sigma_1 : F \rightarrow O(p)$  and  $\sigma_2 : F \rightarrow O(q)$  define free actions on  $S^{p-1}$  and  $S^{q-1}$ ,  $u(\sigma_1, \sigma_2)$  hence defines a free action of  $F$  on  $S^{p+q-1}$ . Using this remark, one could reduce the theorem to the two particular cases  $n = 2$  and  $n = 3$ .

*Proof.* Let  $k \subset \mathbf{R}$  be a real algebraic number field, and  $k' \subset \mathbf{C}$  be a quadratic extension of  $k$ . We assume that  $k' \not\subset \mathbf{R}$ , i.e. that  $k' \otimes_k \mathbf{R} = \mathbf{C}$ . Let  $D$  be a division algebra of dimension  $n^2$  over its center  $k'$ , equipped with an anti-involution  $*$  inducing on  $k'$  the complex conjugation. The  $\mathbf{R}$ -algebra  $D \otimes_k \mathbf{R}$  is a simple algebra over its center  $k' \otimes_k \mathbf{R} = \mathbf{C}$ , hence isomorphic to  $M(n, \mathbf{C})$ . We assume that, for a suitable isomorphism between  $D \otimes_k \mathbf{R}$  and  $M(n, \mathbf{C})$ ,  $*$  becomes transpose conjugate.

In term of an isomorphism as above, the elements  $u$  of  $D$  satisfying  $uu^* = 1$  become unitary matrices. They operate on the unit sphere in  $\mathbf{C}^n$ . Furthermore, if  $u \neq 1$ ,  $u - 1$  is invertible in  $D$  so that the corresponding matrix does not have 1 as an eigenvalue. It hence acts without fixed point on the sphere.

The group  $\Gamma := \{u \in D \mid uu^* = 1\}$  is the group of  $k$ -rational points of a  $k$ -form of the real algebraic group  $U(n)$ . For  $n > 1$ , the perfect subgroup  $SU(n)$

of  $U(n)$  is not trivial, and  $U(n)$  is not solvable. The group  $\Gamma$  is dense in  $U(n)$ : skew adjoints elements of  $D$  are dense in the skew adjoint matrices in  $M(n, \mathbf{C})$ , and the Cayley transform  $t \mapsto \frac{t - 1}{t + 1}$  is an homeomorphism from the space of skew-adjoint matrices in  $M(n, \mathbf{C})$  to an open dense subset of  $U(n)$ , carrying skew adjoint elements of  $D$  into  $\Gamma$ . From this density, it results that, if  $n > 1$ , the linear group  $\Gamma$  is not solvable. By [Tits], it contains a non abelian free subgroup.

It remains to construct pairs  $(D, *)$ . A division algebra  $D$  with center  $k'$  admits an anti-involution  $*$  inducing on  $k'$  the non trivial element  $\text{Gal}(k'/k)$ , if and only if its class  $\text{cl}(D)$  in the Brauer group  $\text{Br}(k')$  of  $k'$  has a trivial image by the norm map  $N_{k'/k} : \text{Br}(k') \rightarrow \text{Br}(k)$ —see Appendix B. Class field theory provides an explicit computation of  $\text{Br}(k)$ , and of  $N_{k'/k}$ , and tells which elements of  $\text{Br}(k')$  come from division algebras. From the explicit description it provides, existence of such  $D$  follows. A direct construction is given in Appendix C. When we choose an isomorphism of  $D \otimes_k \mathbf{R}$  with  $M(n, \mathbf{C})$ , the involution  $*$  becomes adjunction with respect to some hermitian form  $\phi$  on  $\mathbf{C}^n$ , not necessarily positive definite:  $\phi(ax, y) = \phi(x, a^*y)$ . If  $h$  is self adjoint in  $D$ ,  $\text{int}(h^{-1}) \circ *$  is adjunction, with respect to the form  $\phi_h(x, y) = \phi(hx, y)$ . For suitable  $h$ ,  $\phi_h$  is positive definite and  $(D, \text{int}(h^{-1}) \circ *)$  is of the type sought.

### APPENDIX A

Consider  $\phi : S' \cup S'' \rightarrow S - E$  as in the introduction, with  $S'$  and  $S''$  two copies of the sphere  $S$ , and  $\psi : S \rightarrow S'$  the obvious bijection. Consider as in the Schröder-Bernstein theorem the set  $S_e$  of points  $p$  in  $S$  with an even number of ancestors, namely for which there exists an integer  $n \geq 0$  with  $p \in \text{Im}(\phi \circ \psi)^n$  and  $p \notin \text{Im}(\psi \circ (\phi \circ \psi)^n)$ . Consider also the set  $S_0$  of those  $p$  in  $S$  for which there exists  $n \geq 0$  with  $p \in \text{Im}(\psi \circ (\phi \circ \psi)^n)$  and  $p \notin \text{Im}(\phi \circ \psi)^{n+1}$ , and finally the set  $S_\infty$  of those  $p$  such that  $p \in \text{Im}(\phi \circ \psi)^n$  for any  $n \geq 0$ . Consider similarly

$$S' \cup S'' = (S' \cup S'')_e \cup (S' \cup S'')_0 \cup (S' \cup S'')_\infty .$$

Then  $\psi$  induces a bijection from  $S_e \cup S_\infty$  onto  $(S' \cup S'')_0 \cup (S' \cup S'')_\infty$  and  $\phi^{-1}$  from  $S_0$  onto  $(S' \cup S'')_e$ . Combining these two we have a bijection  $\chi : S \rightarrow S' \cup S''$  and a partition of  $S$  into finitely many pieces, the restriction of  $\chi$  to each of these being a rotation.

## APPENDIX B

Let  $K$  be a separable quadratic extension of a field  $k$ . We denote  $x \mapsto \bar{x}$  the non trivial element  $\text{Gal}(K/k)$ . Let  $D$  be a simple algebra with dimension  $n^2$  over its center  $K$ . We will check the criterion of the text, for the existence of an involution of the second kind on  $D$ , i.e. of an anti-involution  $*$  of  $D$ , inducing  $x \mapsto \bar{x}$  on  $K$ . The criterion is that  $N_{K/k} \text{cl}(D) = 0$  in  $\text{Br}(k)$ .

Let us localize, for the étale topology, over  $\text{Spec}(k)$ . This means making large enough étale extensions of scalars  $\otimes_k k'$ , and keeping track of the functoriality in  $k'$ . The field  $K$  becomes the separable quadratic extension  $K' = K \otimes_k k'$  of  $k'$ . The algebra  $D$  becomes  $D' = D \otimes_k k'$ , and is of the form  $D' = \text{End}_{K'}(V')$ , for  $V'$  a free module  $K'$ . The module  $V'$  is not determined uniquely by  $D'$ , only up to homotheties (the corresponding projective space is uniquely determined).

For any  $K$ -module  $M$ , let  $M^-$  be the module deduced from  $M$  by the extension of scalars  $\bar{\phantom{x}} : K \rightarrow K$ , i.e. the module, unique up to unique isomorphism, provided with an anti-linear isomorphism  $x \mapsto \bar{x} : M \xrightarrow{\sim} M^-$ . Similarly for  $K'$ -modules. If  $D' = \text{End}(V')$ , then  $D'^- = \text{End}(V'^-)$ , and

$$(D \otimes_k D^-)' = \text{End}(V' \otimes V'^-).$$

Let  $W'$  be the fixed subspace of the anti-linear automorphism of  $V' \otimes V'^-$  defined by  $v \otimes \bar{w} \mapsto w \otimes \bar{v}$ . It is the space of Hermitian forms on the dual of  $V'$ . One has  $W' \otimes_{k'} K' = V' \otimes V'^-$ . If  $D_1 \subset D \otimes_k D^-$  is the fixed subspace of the anti-linear automorphism of  $D \otimes_k D^-$  defined by  $x \otimes \bar{y} \mapsto y \otimes \bar{x}$ , then  $D'_1$  is the  $k'$ -form of the  $K'$ -algebra  $(D \otimes_k D^-)' = \text{End}(V' \otimes V'^-)$  deduced from the  $k'$ -form  $W'$  of the  $K'$ -module  $V' \otimes V'^- : D'_1 = \text{End}_{k'}(W')$ .

Involutions of the second kind on  $D'$  correspond one to one to non degenerate Hermitian forms on  $V'$ , taken up to a factor (in  $k'^*$ ). Those, in turn, by the “dual form” construction, correspond to “non degenerate” elements of  $W'$ . Again, one has to take them up to a factor. The projective space  $\mathbf{P}(W')$  over  $k'$  is determined up to unique isomorphism by  $D'$ . It is hence (this is the point of localisation) defined over  $k : \mathbf{P}(W') = P \otimes_k k'$ , functorially in  $k'$ . The  $k$ -points of  $P$  (rather, the non degenerate points) parametrize the involutions of the second kind on  $D$ .

The functorial isomorphism  $D'_1 = \text{End}_{k'}(W')$  shows that  $P$  is the form of projective space (Severi-Brauer variety) attached to  $D_1$ . It has a rational point, and is then the ordinary projective space, if and only if  $D_1$  is a matrix algebra.

This shows that  $D$  has involutions of the second kind if and only if the class of  $D_1$  in  $\text{Br}(k)$  is trivial. This class is the required norm  $N_{K/k}(\text{cl}(D))$ . In the localization spirit, this can be deduced from the fact that the homothety by  $\lambda \in K'^*$  of  $V'$  induces on  $W'$  the homothety by  $N_{K'/k'}(\lambda) \in k'^*$ .

### APPENDIX C

For  $n \geq 3$ , examples can be obtained as follows: take  $k' = \mathbf{Q}[\zeta]$ , with  $\zeta = \exp(2\pi i/n)$ , and  $k = k' \cap \mathbf{R}$ . Fix  $a, b \in k'^*$  and let  $D$  be the  $k'$ -algebra generated by  $X, Y$ , subject to

$$\begin{aligned} X^n &= a, & Y^n &= b \\ XY &= \zeta YX. \end{aligned}$$

It admits the anti-involution  $*$ , inducing complex conjugation on  $k'$ , defined by  $\zeta^* = \zeta^{-1}$ ,  $X^* = X$ ,  $Y^* = Y$ . The algebra  $D$  is of the type we require, provided it is a division algebra. This happens already with  $a, b \in \mathbf{Z}$ : take for  $a$  a prime congruent to 1 mod  $n$ , and for  $b$  an integer whose residue mod  $a$  has in the cyclic group of order  $n$   $(\mathbf{Z}/(a))^*/(\mathbf{Z}/(a))^{*n}$  an image of exact order  $n$ . For instance  $n = 3$ ,  $a = 7$ ,  $b = 2$ . For  $n = 2$ , one proceeds similarly with  $k' = \mathbf{Q}[i]$ ,  $\zeta = -1$ ,  $a$  congruent to 1 mod 4 and  $b$  not a square mod  $a$ . For instance,  $a = 5$ , and  $b = 2$ . In each case, the assumption on  $a$  ensures that  $k'$  embed in the  $a$ -adic completion  $\mathbf{Q}_a$  of  $\mathbf{Q}$ , and the fact that  $D$  is a division algebra can be seen locally at  $a$ :  $D \otimes_{k'} \mathbf{Q}_a$  is a division algebra with center  $\mathbf{Q}_a$ .

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