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**Autor:** de la Harpe, Pierre

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homomorphism  $PGL(n, \mathbf{C}) \rightarrow PGL(N, \mathbf{C})$ . Then  $\lambda_k(\tilde{\gamma})$  has eigenvalues  $v_1, \dots, v_N$  with  $|v_1| > |v_j|$  for  $j = 2, \dots, N$ . By lemma 3, there exists a  $\lambda_k(\tilde{\Gamma})$ -irreducible subspace  $W_0$  of  $\mathbf{C}^N$ , associated to a representation  $\sigma_0: \tilde{\Gamma} \rightarrow GL(W_0)$ , such that  $v_1$  is an eigenvalue of  $\sigma_0(\tilde{\gamma})$ . As the  $Z$ -closure  $\tilde{G}$  of  $\tilde{\Gamma}$  in  $SL(n, \mathbf{C})$  is semi-simple, the group  $\tilde{G}$  is perfect and  $\sigma_0(\tilde{\Gamma})$  lies in  $SL(W_0)$ . As  $|v_1| > 1$ , one has  $\dim_{\mathbf{C}} W_0 \geq 2$ .

Thus one may assume from the start that  $\Gamma$  contains a sharp semi-simple element, and indeed by lemmas 1 and 2 two very sharp elements in general position. The conclusion follows as in case 2 of the proof of the proposition in section 4.  $\square$

Now lemma 1 remains true without the hypothesis “semi-simple”. This has been announced by Y. Guivarch, who uses ideas of H. Fürstenberg to show the following: given an appropriate subset  $S$  of  $\Gamma$  containing a sharp element, then almost any “long” word in the letters of  $S$  is very sharp. Using this, one may replace (ii) in the theorem above by the following a priori weaker hypothesis

(ii')  $\Gamma$  is not relatively compact.

Then, one first checks as for theorem 2 of section 4 that  $\Gamma$  contains hyperbolic elements; one concludes as in the previous proof, with Guivarch’s version of lemma 1.

For subgroups of  $PU(n)$ , one may repeat the discussion at the end of section 4.

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Pierre de la Harpe  
Section de mathématiques  
C.P. 240  
CH-1211 Genève 24