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$X^d F\left(\frac{1}{2}X + \frac{1}{2}X^{-1}\right)$, which is of degree $2d$ in $Z[X]$, so that $2d \geqslant \varphi(q)$. If $q \in \{1, 2, 3, 4, 6\}$, one checks easily that $\exp\left(i2\pi \frac{p}{q}\right) \neq \frac{3 + 4i}{5}$. If $q = 5$ or if $q \geqslant 7$, then $\varphi(q) > 2$ so that $\cos\left(2\pi \frac{p}{q}\right)$ is not rational. Thus the root of unity μ cannot be equal to $\frac{3 + 4i}{5}$.

5. Some other cases of Tits' theorem

Let n be an integer with $n \geqslant 2$.

Define a subgroup Γ of $GL(n, \mathbf{C})$ [respectively of $PGL(n, \mathbf{C})$] to be *irreducible* if any linear subspace of $\mathbf{C}^n$ [resp. of $P^{n-1}_{\mathbf{C}}$] invariant by Γ is trivial, and *not almost reducible* if any subgroup of Γ of finite index is irreducible. When referring to the *Zariski* topology on $PGL(n, \mathbf{C})$, we use below the letter Z.

Reduction. Tits' theorem for complex linear groups is equivalent to the following statements (one for each $n \geqslant 2$):

Let Γ be a subgroup of $PGL(n, \mathbf{C})$ which is not almost solvable. Assume that

(i) is not almost reducible;

(ii) the Z-closure G of Γ in $PGL(n, \mathbf{C})$ is Z-connected. Then Γ contains free groups.

That one may assume (i) without loss of generality is an easy exercise on reducibility, and one may assume (ii) because the Z-closure of any subgroup of $PGL(n, \mathbf{C})$ has finitely many Z-connected components. (The hypothesis of the reduced statement are redundant: (i) and (ii) imply by Lie's theorem that G is not solvable, so that Γ is not almost solvable!)

Now let $g \in PGL(n, \mathbf{C})$ and choose a representative $\tilde{g} \in GL(n, \mathbf{C})$ of g. Let us define g to be

elliptic if $\tilde{g}$ is semi-simple with all eigenvalues of equal moduli,

parabolic if $\tilde{g}$ is not semi-simple and has all its eigenvalues of equal moduli,

hyperbolic if $\tilde{g}$ has at least two eigenvalues of distinct moduli.

These definitions are obviously independent on the choice of $\tilde{g}$. They generalize those of section 3 as follows from [Gr]. The meaning of "hyperbolic" fits with current use in dynamical systems theory (see e.g. definition 5.1 in [Sh]).

Let g be hyperbolic and let $\tilde{g}$ be as above. Let $\tilde{A}(g)$ [respectively $\tilde{A}'(g)$] be the direct sum of the nilspaces of $\tilde{g}$ corresponding to all eigenvalues of maximal modulus [resp. to all other eigenvalues] of $\tilde{g}$. Let $A(g)$ [resp. $A'(g)$] be the canonical image of $\tilde{A}(g) - \{0\}$ [resp. $\tilde{A}'(g) - \{0\}$] in $\mathbf{P} = P_{\mathbf{C}}^{n-1}$. Then $A(g) \cap A'(g) = \emptyset$ and the smallest linear subspace of $\mathbf{P}$ containing both $A(g)$ and $A'(g)$ is $\mathbf{P}$ itself. Tits calls $A(g)$ [resp. $A(g^{-1})$] the *attracting space* [resp. *repulsing space*] of g. We say that g is *sharp* if $A(g)$ is a point and that g is *very sharp* if both $A(g)$ and $A(g^{-1})$ are points. For each $k \in \{1, 2, ..., n-1\}$, the fundamental representation of $GL(n, C)$ in $\wedge^k \mathbf{C}^n$ induces an injection

$$\lambda_k : PGL(n, \mathbf{C}) \to PGL(\tbinom{n}{k}, \mathbf{C});$$

as g is hyperbolic, $\lambda_k(g)$ is sharp for some k. We also say that two hyperbolic elements $g, h \in PGL(n, \mathbf{C})$ are in *general position* if

$$A(g) \cup A(g^{-1}) \subset \mathbf{P} - \{A'(h) \cup A'(h^{-1})\}$$
$$A(h) \cup A(h^{-1}) \subset \mathbf{P} - \{A'(g) \cup A'(g^{-1})\} .$$

Observe that any hyperbolic element of $PGL(2, \mathbf{C})$ is very sharp, and that two hyperbolic elements of $PGL(2, \mathbf{C})$ are in general position if and only if they do not have any common fixed point on $\mathbf{S}^2$.

Recall that an element of $PGL(n, \mathbf{C})$ is *semi-simple* if its inverse image in $GL(n, \mathbf{C})$ contains diagonalisable matrices.

LEMMA 1. *Let* Γ *be an irreducible subgroup of* $PGL(n, \mathbf{C})$ *having a Z-connected Z-closure. If* Γ *contains a sharp semi-simple element* g, *then* Γ *contains a very sharp element.*

About the proof. Let $\tilde{g} \in GL(n, \mathbf{C})$ be some representative of g having an eigenvalue of "large" modulus and all other eigenvalues with moduli "near" 1. For suitable $h, u \in \Gamma$ and for $j \in N$ large enough, one may hope that $g^{-j}hg^jh^{-1}u$ has a representative in $GL(n, \mathbf{C})$ with one eigenvalue of very large modulus (look at $hg^jh^{-1}u$), one eigenvalue of very small modulus (look at g^{-j}), and other eigenvalues of moduli "near" 1. Section 3 of [T] shows that this hope is realistic. (See also below, after the theorem.) □

LEMMA 2. *Let* Γ *be an irreducible subgroup of* $PGL(n, \mathbf{C})$ *having a Z-connected Z-closure. If* Γ *contains a very sharp element, then* Γ *contains two very sharp elements in general position.*

Proof. Let P_1, P_2 be two linear subspaces of $\mathbf{P}$ with $P_1 \neq \emptyset$ and $P_2 \neq \mathbf{P}$. Then $\{ x \in G \mid x(P_1) \not\subset P_2 \}$ is obviously a Z-open subset of G. It is not empty:

Choose indeed $p \in P_1$; then the subspace of **P** spanned by the orbit Gp is stable under G and must therefore coincide with **P**; hence there exists $x \in G$ with $x(p) \notin P_2$ and, a fortiori, $x(P_1) \not\subset P_2$.

Let g be a very sharp element in Γ. It follows from above that

$$X = \left\{ x \in G \;\middle|\; \begin{array}{l} A(g) \text{ and } A(g^{-1}) \text{ are not contained in any of } xA'(g)\,, \\ xA'(g^{-1}),\, x^{-1}A'(g),\, x^{-1}A'(g^{-1}) \end{array} \right\}$$

is a non empty Z-open subset of G. Let $y \in X \cap \Gamma$. Then g and ygy^{-1} are both very sharp and are in general position. □

For the next lemma, we choose as above k with $1 \leqslant k \leqslant n-1$ and we consider the k^{th} fundamental representation $\lambda_k: SL(n, \mathbf{C}) \to SL(\binom{n}{k}, \mathbf{C})$ of $SL(n, \mathbf{C})$.

LEMMA. *Let Γ be a group and let $\rho: \Gamma \to SL(n, \mathbf{C})$ be an irreducible representation. Then the Z-closure G of $\rho(\Gamma)$ in $SL(n, \mathbf{C})$ is semi-simple and the representation $\sigma = \lambda_k\rho: \Gamma \to SL(\binom{n}{k}, \mathbf{C})$ is completely reducible.*

Proof. We show first that G is semi-simple. Consider the solvable radical R of G. By Lie's theorem, there exists an eigenvector for R, namely there exist $v \in \mathbf{C}^n - \{0\}$ and $\alpha \in \operatorname{Hom}(R, \mathbf{C}^*)$ with $r(v) = \alpha(r)v$ for all $r \in R$. As R is normal in G, any vector $g(v)$ ($g \in G$) is also an eigenvector for R. By irreductibility, any vector in $\mathbf{C}^n$ is also an eigenvector, so that R is made up of dilations. But R is connected and is in $SL(n, \mathbf{C})$, so that $R = 1$.

Now $\lambda_k: G \to SL(\binom{n}{k}, \mathbf{C})$ is completely reducible; denote by $\lambda_{k,j}: G \to SL(W_j)$ the components of a decomposition $\lambda_k = \bigoplus_{j \in J} \lambda_{k,j}$ and define $\sigma_j = \lambda_{k,j}\rho$ ($j \in J$). One has clearly $\sigma = \bigoplus_{j \in J} \sigma_j$, and each $\sigma_j: \Gamma \to SL(W_j)$ is irreducible (this because $\lambda_{k,j}$ is irreducible and by Schur's lemma). □

THEOREM. *Let Γ be a subgroup of $PGL(n, \mathbf{C})$ and assume*

(i) Γ *is neither almost solvable nor almost reducible,*

(ii) Γ *contains a semi-simple hyperbolic element.*

Then Γ contains free groups.

Proof. As one may consider instead of Γ a subgroup of finite index, there is no loss of generality if we assume that the Z-closure of Γ is Z-connected. We denote by $\tilde{\Gamma}$ the inverse image of Γ in $SL(n, \mathbf{C})$. By (ii), there exists $k \in \{1, \ldots, n-1\}$ and a semi-simple element $\tilde{\gamma} \in \tilde{\Gamma}$ having eigenvalues $\mu_1, \ldots, \mu_n$ with $|\mu_1| = \ldots = |\mu_k| > |\mu_j|$ for $j = k+1, \ldots, n$. Let $N = \binom{n}{k}$, and denote by λ_k both the fundamental representation $GL(n, \mathbf{C}) \to GL(N, \mathbf{C})$ and the induced homomorphism $PGL(n, \mathbf{C}) \to PGL(N, \mathbf{C})$. Then $\lambda_k(\tilde{\gamma})$ has eigenvalues $\nu_1, ..., \nu_N$ with $|\nu_1| > |\nu_j|$ for $j = 2, ..., N$. By lemma 3, there exists a $\lambda_k(\tilde{\Gamma})$-irreducible subspace W_0 of $\mathbf{C}^N$, associated to a representation $\sigma_0 : \tilde{\Gamma} \to GL(W_0)$, such that ν_1 is an eigenvalue of $\sigma_0(\tilde{\gamma})$. As the Z-closure $\tilde{G}$ of $\tilde{\Gamma}$ in $SL(n, \mathbf{C})$ is semi-simple, the group $\tilde{G}$ is perfect and $\sigma_0(\tilde{\Gamma})$ lies in $SL(W_0)$. As $|\nu_1| > 1$, one has $\dim_{\mathbf{C}} W_0 \geqslant 2$.

Thus one may assume from the start that Γ contains a sharp semi-simple element, and indeed by lemmas 1 and 2 two very sharp elements in general position. The conclusion follows as in case 2 of the proof of the proposition in section 4. □

Now lemma 1 remains true without the hypothesis "semi-simple". This has been announced by Y. Guivarch', who uses ideas of H. Fürstenberg to show the following: given an appropriate subset S of Γ containing a sharp element, then almost any "long" word in the letters of S is very sharp. Using this, one may replace (ii) in the theorem above by the following a priori weaker hypothesis

(ii′) Γ is not relatively compact.

Then, one first checks as for theorem 2 of section 4 that Γ contains hyperbolic elements; one concludes as in the previous proof, with Guivarch's version of lemma 1.

For subgroups of $PU(n)$, one may repeat the discussion at the end of section 4.

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