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$X^d F \left(\frac{1}{2} X + \frac{1}{2} X^{-1} \right)$, which is of degree $2d$ in $\mathbb{Z}[X]$, so that $2d \geq \varphi(q)$. If $q \in \{1, 2, 3, 4, 6\}$, one checks easily that $\exp\left(i2\pi \frac{p}{q}\right) \neq \frac{3+4i}{5}$. If $q = 5$ or if $q \geq 7$, then $\varphi(q) > 2$ so that $\cos\left(2\pi \frac{p}{q}\right)$ is not rational. Thus the root of unity μ cannot be equal to $\frac{3+4i}{5}$.

5. SOME OTHER CASES OF TITS' THEOREM

Let n be an integer with $n \geq 2$.

Define a subgroup Γ of $GL(n, \mathbf{C})$ [respectively of $PGL(n, \mathbf{C})$] to be *irreducible* if any linear subspace of \mathbf{C}^n [resp. of $P_{\mathbf{C}}^{n-1}$] invariant by Γ is trivial, and *not almost reducible* if any subgroup of Γ of finite index is irreducible. When referring to the Zariski topology on $PGL(n, \mathbf{C})$, we use below the letter Z .

Reduction. Tits' theorem for complex linear groups is equivalent to the following statements (one for each $n \geq 2$):

- Let Γ be a subgroup of $PGL(n, \mathbf{C})$ which is not almost solvable. Assume that
- (i) is not almost reducible;
- (ii) the Z -closure G of Γ in $PGL(n, \mathbf{C})$ is Z -connected. Then Γ contains free groups.

That one may assume (i) without loss of generality is an easy exercise on reducibility, and one may assume (ii) because the Z -closure of any subgroup of $PGL(n, \mathbf{C})$ has finitely many Z -connected components. (The hypothesis of the reduced statement are redundant: (i) and (ii) imply by Lie's theorem that G is not solvable, so that Γ is not almost solvable!)

Now let $g \in PGL(n, \mathbf{C})$ and choose a representative $\tilde{g} \in GL(n, \mathbf{C})$ of g . Let us define g to be

elliptic if \tilde{g} is semi-simple with all eigenvalues of equal moduli,

parabolic if \tilde{g} is not semi-simple and has all its eigenvalues of equal moduli,

hyperbolic if \tilde{g} has at least two eigenvalues of distinct moduli.

These definitions are obviously independent on the choice of \tilde{g} . They generalize those of section 3 as follows from [Gr]. The meaning of "hyperbolic" fits with current use in dynamical systems theory (see e.g. definition 5.1 in [Sh]).

Let g be hyperbolic and let \tilde{g} be as above. Let $\tilde{A}(g)$ [respectively $\tilde{A}'(g)$] be the direct sum of the nilspaces of \tilde{g} corresponding to all eigenvalues of maximal modulus [resp. to all other eigenvalues] of \tilde{g} . Let $A(g)$ [resp. $A'(g)$] be the canonical image of $\tilde{A}(g) - \{0\}$ [resp. $\tilde{A}'(g) - \{0\}$] in $\mathbf{P} = P_{\mathbf{C}}^{n-1}$. Then $A(g) \cap A'(g) = \emptyset$ and the smallest linear subspace of \mathbf{P} containing both $A(g)$ and $A'(g)$ is \mathbf{P} itself. Tits calls $A(g)$ [resp. $A(g^{-1})$] the *attracting space* [resp. *repulsing space*] of g . We say that g is *sharp* if $A(g)$ is a point and that g is *very sharp* if both $A(g)$ and $A(g^{-1})$ are points. For each $k \in \{1, 2, \dots, n-1\}$, the fundamental representation of $GL(n, \mathbf{C})$ in $\wedge^k \mathbf{C}^n$ induces an injection

$$\lambda_k: PGL(n, \mathbf{C}) \rightarrow PGL(\binom{n}{k}, \mathbf{C});$$

as g is hyperbolic, $\lambda_k(g)$ is sharp for some k . We also say that two hyperbolic elements $g, h \in PGL(n, \mathbf{C})$ are in *general position* if

$$\begin{aligned} A(g) \cup A(g^{-1}) &\subset \mathbf{P} - \{A'(h) \cup A'(h^{-1})\} \\ A(h) \cup A(h^{-1}) &\subset \mathbf{P} - \{A'(g) \cup A'(g^{-1})\}. \end{aligned}$$

Observe that any hyperbolic element of $PGL(2, \mathbf{C})$ is very sharp, and that two hyperbolic elements of $PGL(2, \mathbf{C})$ are in general position if and only if they do not have any common fixed point on S^2 .

Recall that an element of $PGL(n, \mathbf{C})$ is *semi-simple* if its inverse image in $GL(n, \mathbf{C})$ contains diagonalisable matrices.

LEMMA 1. *Let Γ be an irreducible subgroup of $PGL(n, \mathbf{C})$ having a Z -connected Z -closure. If Γ contains a sharp semi-simple element g , then Γ contains a very sharp element.*

About the proof. Let $\tilde{g} \in GL(n, \mathbf{C})$ be some representative of g having an eigenvalue of “large” modulus and all other eigenvalues with moduli “near” 1. For suitable $h, u \in \Gamma$ and for $j \in \mathbb{N}$ large enough, one may hope that $g^{-j}hg^jh^{-1}u$ has a representative in $GL(n, \mathbf{C})$ with one eigenvalue of very large modulus (look at $hg^jh^{-1}u$), one eigenvalue of very small modulus (look at g^{-j}), and other eigenvalues of moduli “near” 1. Section 3 of [T] shows that this hope is realistic. (See also below, after the theorem.) \square

LEMMA 2. *Let Γ be an irreducible subgroup of $PGL(n, \mathbf{C})$ having a Z -connected Z -closure. If Γ contains a very sharp element, then Γ contains two very sharp elements in general position.*

Proof. Let P_1, P_2 be two linear subspaces of \mathbf{P} with $P_1 \neq \emptyset$ and $P_2 \neq \mathbf{P}$. Then $\{x \in G \mid x(P_1) \notin P_2\}$ is obviously a Z -open subset of G . It is not empty:

Choose indeed $p \in P_1$; then the subspace of \mathbf{P} spanned by the orbit Gp is stable under G and must therefore coincide with \mathbf{P} ; hence there exists $x \in G$ with $x(p) \notin P_2$ and, a fortiori, $x(P_1) \notin P_2$.

Let g be a very sharp element in Γ . It follows from above that

$$X = \left\{ x \in G \mid \begin{array}{l} A(g) \text{ and } A(g^{-1}) \text{ are not contained in any of } xA'(g), \\ xA'(g^{-1}), x^{-1}A'(g), x^{-1}A'(g^{-1}) \end{array} \right\}$$

is a non empty Z -open subset of G . Let $y \in X \cap \Gamma$. Then g and ygy^{-1} are both very sharp and are in general position. \square

For the next lemma, we choose as above k with $1 \leq k \leq n-1$ and we consider the k^{th} fundamental representation $\lambda_k: SL(n, \mathbf{C}) \rightarrow SL(\binom{n}{k}, \mathbf{C})$ of $SL(n, \mathbf{C})$.

LEMMA. *Let Γ be a group and let $\rho: \Gamma \rightarrow SL(n, \mathbf{C})$ be an irreducible representation. Then the Z -closure G of $\rho(\Gamma)$ in $SL(n, \mathbf{C})$ is semi-simple and the representation $\sigma = \lambda_k \rho: \Gamma \rightarrow SL(\binom{n}{k}, \mathbf{C})$ is completely reducible.*

Proof. We show first that G is semi-simple. Consider the solvable radical R of G . By Lie's theorem, there exists an eigenvector for R , namely there exist $v \in \mathbf{C}^n - \{0\}$ and $\alpha \in \text{Hom}(R, \mathbf{C}^*)$ with $r(v) = \alpha(r)v$ for all $r \in R$. As R is normal in G , any vector $g(v)$ ($g \in G$) is also an eigenvector for R . By irreducibility, any vector in \mathbf{C}^n is also an eigenvector, so that R is made up of dilations. But R is connected and is in $SL(n, \mathbf{C})$, so that $R = 1$.

Now $\lambda_k: G \rightarrow SL(\binom{n}{k}, \mathbf{C})$ is completely reducible; denote by $\lambda_{k,j}: G \rightarrow SL(W_j)$ the components of a decomposition $\lambda_k = \bigoplus_{j \in J} \lambda_{k,j}$ and define $\sigma_j = \lambda_{k,j} \rho$ ($j \in J$). One has clearly $\sigma = \bigoplus_{j \in J} \sigma_j$, and each $\sigma_j: \Gamma \rightarrow SL(W_j)$ is irreducible (this because $\lambda_{k,j}$ is irreducible and by Schur's lemma). \square

THEOREM. *Let Γ be a subgroup of $PGL(n, \mathbf{C})$ and assume*

- (i) Γ is neither almost solvable nor almost reducible,
- (ii) Γ contains a semi-simple hyperbolic element.

Then Γ contains free groups.

Proof. As one may consider instead of Γ a subgroup of finite index, there is no loss of generality if we assume that the Z -closure of Γ is Z -connected. We denote by $\tilde{\Gamma}$ the inverse image of Γ in $SL(n, \mathbf{C})$. By (ii), there exists $k \in \{1, \dots, n-1\}$ and a semi-simple element $\tilde{\gamma} \in \tilde{\Gamma}$ having eigenvalues μ_1, \dots, μ_n with $|\mu_1| = \dots = |\mu_k| > |\mu_j|$ for $j = k+1, \dots, n$. Let $N = \binom{n}{k}$, and denote by λ_k both the fundamental representation $GL(n, \mathbf{C}) \rightarrow GL(N, \mathbf{C})$ and the induced

homomorphism $PGL(n, \mathbf{C}) \rightarrow PGL(N, \mathbf{C})$. Then $\lambda_k(\tilde{\gamma})$ has eigenvalues v_1, \dots, v_N with $|v_1| > |v_j|$ for $j = 2, \dots, N$. By lemma 3, there exists a $\lambda_k(\tilde{\Gamma})$ -irreducible subspace W_0 of \mathbf{C}^N , associated to a representation $\sigma_0: \tilde{\Gamma} \rightarrow GL(W_0)$, such that v_1 is an eigenvalue of $\sigma_0(\tilde{\gamma})$. As the \mathbf{Z} -closure \tilde{G} of $\tilde{\Gamma}$ in $SL(n, \mathbf{C})$ is semi-simple, the group \tilde{G} is perfect and $\sigma_0(\tilde{\Gamma})$ lies in $SL(W_0)$. As $|v_1| > 1$, one has $\dim_{\mathbf{C}} W_0 \geq 2$.

Thus one may assume from the start that Γ contains a sharp semi-simple element, and indeed by lemmas 1 and 2 two very sharp elements in general position. The conclusion follows as in case 2 of the proof of the proposition in section 4. \square

Now lemma 1 remains true without the hypothesis “semi-simple”. This has been announced by Y. Guivarch, who uses ideas of H. Fürstenberg to show the following: given an appropriate subset S of Γ containing a sharp element, then almost any “long” word in the letters of S is very sharp. Using this, one may replace (ii) in the theorem above by the following a priori weaker hypothesis

(ii') Γ is not relatively compact.

Then, one first checks as for theorem 2 of section 4 that Γ contains hyperbolic elements; one concludes as in the previous proof, with Guivarch’s version of lemma 1.

For subgroups of $PU(n)$, one may repeat the discussion at the end of section 4.

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