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 $X^dF\left(\frac{1}{2}X+\frac{1}{2}X^{-1}\right)$ , which is of degree 2d in Z[X], so that  $2d\geqslant \varphi(q)$ . If  $q\in\{1,2,3,4,6\}$ , one checks easily that  $\exp\left(i2\pi\frac{p}{q}\right)\neq\frac{3+4i}{5}$ . If q=5 or if  $q\geqslant 7$ , then  $\varphi(q)>2$  so that  $\cos\left(2\pi\frac{p}{q}\right)$  is not rational. Thus the root of unity  $\mu$  cannot be equal to  $\frac{3+4i}{5}$ .

# 5. Some other cases of Tits' theorem

Let *n* be an integer with  $n \ge 2$ .

Define a subgroup  $\Gamma$  of  $GL(n, \mathbb{C})$  [respectively of  $PGL(n, \mathbb{C})$ ] to be *irreducible* if any linear subspace of  $\mathbb{C}^n$  [resp. of  $P_{\mathbb{C}}^{n-1}$ ] invariant by  $\Gamma$  is trivial, and *not almost* reducible if any subgroup of  $\Gamma$  of finite index is irreducible. When referring to the Zariski topology on  $PGL(n, \mathbb{C})$ , we use below the letter  $\mathbb{Z}$ .

Reduction. Tits' theorem for complex linear groups is equivalent to the following statements (one for each  $n \ge 2$ ):

Let  $\Gamma$  be a subgroup of  $PGL(n, \mathbb{C})$  which is not almost solvable. Assume that

- (i) is not almost reducible;
- (ii) the Z-closure G of  $\Gamma$  in  $PGL(n, \mathbb{C})$  is Z-connected. Then  $\Gamma$  contains free groups.

That one may assume (i) without loss of generality is an easy exercise on reducibility, and one may assume (ii) because the Z-closure of any subgroup of  $PGL(n, \mathbb{C})$  has finitely many Z-connected components. (The hypothesis of the reduced statement are redundant: (i) and (ii) imply by Lie's theorem that G is not solvable, so that  $\Gamma$  is not almost solvable!)

Now let  $g \in PGL(n, \mathbb{C})$  and choose a representative  $\tilde{g} \in GL(n, \mathbb{C})$  of g. Let us define g to be

elliptic if  $\tilde{g}$  is semi-simple with all eigenvalues of equal moduli, parabolic if  $\tilde{g}$  is not semi-simple and has all its eigenvalues of equal moduli, hyperbolic if  $\tilde{g}$  has at least two eigenvalues of distinct moduli.

These definitions are obviously independent on the choice of  $\tilde{g}$ . They generalize those of section 3 as follows from [Gr]. The meaning of "hyperbolic" fits with current use in dynamical systems theory (see e.g. definition 5.1 in [Sh]).

Let g be hyperbolic and let  $\tilde{g}$  be as above. Let  $\tilde{A}(g)$  [respectively  $\tilde{A}'(g)$ ] be the direct sum of the nilspaces of  $\tilde{g}$  corresponding to all eigenvalues of maximal modulus [resp. to all other eigenvalues] of  $\tilde{g}$ . Let A(g) [resp. A'(g)] be the canonical image of  $\tilde{A}(g) - \{0\}$  [resp.  $\tilde{A}'(g) - \{0\}$ ] in  $\mathbf{P} = P_{\mathbf{C}}^{n-1}$ . Then  $A(g) \cap A'(g) = \emptyset$  and the smallest linear subspace of  $\mathbf{P}$  containing both A(g) and A'(g) is  $\mathbf{P}$  itself. Tits calls A(g) [resp.  $A(g^{-1})$ ] the attracting space [resp. repulsing space] of g. We say that g is sharp if A(g) is a point and that g is very sharp if both A(g) and  $A(g^{-1})$  are points. For each  $k \in \{1, 2, ..., n-1\}$ , the fundamental representation of GL(n, C) in  $\wedge^k \mathbf{C}^n$  induces an injection

$$\lambda_k: PGL(n, \mathbb{C}) \to PGL(\binom{n}{k}, \mathbb{C});$$

as g is hyperbolic,  $\lambda_k(g)$  is sharp for some k. We also say that two hyperbolic elements  $g, h \in PGL(n, \mathbb{C})$  are in general position if

$$A(g) \cup A(g^{-1}) \subset \mathbf{P} - \{A'(h) \cup A'(h^{-1})\}\$$
  
 $A(h) \cup A(h^{-1}) \subset \mathbf{P} - \{A'(g) \cup A'(g^{-1})\}\$ .

Observe that any hyperbolic element of  $PGL(2, \mathbb{C})$  is very sharp, and that two hyperbolic elements of  $PGL(2, \mathbb{C})$  are in general position if and only if they do not have any common fixed point on  $\mathbb{S}^2$ .

Recall that an element of  $PGL(n, \mathbb{C})$  is semi-simple if its inverse image in  $GL(n, \mathbb{C})$  contains diagonalisable matrices.

LEMMA 1. Let  $\Gamma$  be an irreducible subgroup of  $PGL(n, \mathbb{C})$  having a Z-connected Z-closure. If  $\Gamma$  contains a sharp semi-simple element g, then  $\Gamma$  contains a very sharp element.

About the proof. Let  $\tilde{g} \in GL(n, \mathbb{C})$  be some representative of g having an eigenvalue of "large" modulus and all other eigenvalues with moduli "near" 1. For suitable  $h, u \in \Gamma$  and for  $j \in N$  large enough, one may hope that  $g^{-j}hg^{j}h^{-1}u$  has a representative in  $GL(n, \mathbb{C})$  with one eigenvalue of very large modulus (look at  $hg^{j}h^{-1}u$ ), one eigenvalue of very small modulus (look at  $g^{-j}$ ), and other eigenvalues of moduli "near" 1. Section 3 of [T] shows that this hope is realistic. (See also below, after the theorem.)

LEMMA 2. Let  $\Gamma$  be an irreducible subgroup of  $PGL(n, \mathbb{C})$  having a Z-connected Z-closure. If  $\Gamma$  contains a very sharp element, then  $\Gamma$  contains two very sharp elements in general position.

*Proof.* Let  $P_1$ ,  $P_2$  be two linear subspaces of **P** with  $P_1 \neq \emptyset$  and  $P_2 \neq \mathbf{P}$ . Then  $\{x \in G \mid x(P_1) \neq P_2\}$  is obviously a Z-open subset of G. It is not empty:

Choose indeed  $p \in P_1$ ; then the subspace of **P** spanned by the orbit Gp is stable under G and must therefore coincide with **P**; hence there exists  $x \in G$  with  $x(p) \notin P_2$  and, a fortiori,  $x(P_1) \notin P_2$ .

Let g be a very sharp element in  $\Gamma$ . It follows from above that

$$X = \left\{ x \in G \middle| \begin{array}{l} A(g) \text{ and } A(g^{-1}) \text{ are not contained in any of } xA'(g), \\ xA'(g^{-1}), x^{-1}A'(g), x^{-1}A'(g^{-1}) \end{array} \right\}$$

is a non empty Z-open subset of G. Let  $y \in X \cap \Gamma$ . Then g and  $ygy^{-1}$  are both very sharp and are in general position.

For the next lemma, we choose as above k with  $1 \le k \le n-1$  and we consider the  $k^{\text{th}}$  fundamental representation  $\lambda_k : SL(n, \mathbb{C}) \to SL(\binom{n}{k}, \mathbb{C})$  of  $SL(n, \mathbb{C})$ .

LEMMA. Let  $\Gamma$  be a group and let  $\rho: \Gamma \to SL(n, \mathbb{C})$  be an irreducible representation. Then the Z-closure G of  $\rho(\Gamma)$  in  $SL(n, \mathbb{C})$  is semi-simple and the representation  $\sigma = \lambda_k \rho: \Gamma \to SL(\binom{n}{k}, \mathbb{C})$  is completely reducible.

*Proof.* We show first that G is semi-simple. Consider the solvable radical R of G. By Lie's theorem, there exists an eigenvector for R, namely there exist  $v \in \mathbb{C}^n - \{0\}$  and  $\alpha \in \text{Hom}(R, \mathbb{C}^*)$  with  $r(v) = \alpha(r)v$  for all  $r \in R$ . As R is normal in G, any vector g(v) ( $g \in G$ ) is also an eigenvector for R. By irreductibility, any vector in  $\mathbb{C}^n$  is also an eigenvector, so that R is made up of dilations. But R is connected and is in  $SL(n, \mathbb{C})$ , so that R = 1.

Now  $\lambda_k: G \to SL(\binom{n}{k}, \mathbb{C})$  is completely reducible; denote by  $\lambda_{k,j}: G \to SL(W_j)$  the components of a decomposition  $\lambda_k = \bigoplus_{j \in J} \lambda_{k,j}$  and define  $\sigma_j = \lambda_{k,j} \rho$   $(j \in J)$ . One has clearly  $\sigma = \bigoplus_{j \in J} \sigma_j$ , and each  $\sigma_j: \Gamma \to SL(W_j)$  is irreducible (this because  $\lambda_{k,j}$  is irreducible and by Schur's lemma).

Theorem. Let  $\Gamma$  be a subgroup of  $PGL(n, \mathbb{C})$  and assume

- (i)  $\Gamma$  is neither almost solvable nor almost reducible,
- (ii)  $\Gamma$  contains a semi-simple hyperbolic element.

Then  $\Gamma$  contains free groups.

*Proof.* As one may consider instead of  $\Gamma$  a subgroup of finite index, there is no loss of generality if we assume that the Z-closure of  $\Gamma$  is Z-connected. We denote by  $\widetilde{\Gamma}$  the inverse image of  $\Gamma$  in  $SL(n, \mathbb{C})$ . By (ii), there exists  $k \in \{1, ..., n-1\}$  and a semi-simple element  $\widetilde{\gamma} \in \widetilde{\Gamma}$  having eigenvalues  $\mu_1, ..., \mu_n$  with  $|\mu_1| = ...$   $= |\mu_k| > |\mu_j|$  for j = k + 1, ..., n. Let  $N = \binom{n}{k}$ , and denote by  $\lambda_k$  both the fundamental representation  $GL(n, \mathbb{C}) \to GL(N, \mathbb{C})$  and the induced

homomorphism  $PGL(n, \mathbb{C}) \to PGL(N, \mathbb{C})$ . Then  $\lambda_k(\tilde{\gamma})$  has eigenvalues  $v_1, ..., v_N$  with  $|v_1| > |v_j|$  for j = 2, ..., N. By lemma 3, there exists a  $\lambda_k(\tilde{\Gamma})$ -irreducible subspace  $W_0$  of  $\mathbb{C}^N$ , associated to a representation  $\sigma_0 \colon \tilde{\Gamma} \to GL(W_0)$ , such that  $v_1$  is an eigenvalue of  $\sigma_0(\tilde{\gamma})$ . As the Z-closure  $\tilde{G}$  of  $\tilde{\Gamma}$  in  $SL(n, \mathbb{C})$  is semi-simple, the group  $\tilde{G}$  is perfect and  $\sigma_0(\tilde{\Gamma})$  lies in  $SL(W_0)$ . As  $|v_1| > 1$ , one has  $\dim_{\mathbb{C}} W_0 \ge 2$ .

Thus one may assume from the start that  $\Gamma$  contains a sharp semi-simple element, and indeed by lemmas 1 and 2 two very sharp elements in general position. The conclusion follows as in case 2 of the proof of the proposition in section 4.

Now lemma 1 remains true without the hypothesis "semi-simple". This has been announced by Y. Guivarch', who uses ideas of H. Fürstenberg to show the following: given an appropriate subset S of  $\Gamma$  containing a sharp element, then almost any "long" word in the letters of S is very sharp. Using this, one may replace (ii) in the theorem above by the following a priori weaker hypothesis

(ii')  $\Gamma$  is not relatively compact.

Then, one first checks as for theorem 2 of section 4 that  $\Gamma$  contains hyperbolic elements; one concludes as in the previous proof, with Guivarch's version of lemma 1.

For subgroups of PU(n), one may repeat the discussion at the end of section 4.

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