5. SOME OTHER CASES OF TITS' THEOREM

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **29 (1983)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **27.04.2024**

Nutzungsbedingungen
Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss
Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

$X^d F\left(\frac{1}{2}X + \frac{1}{2}X^{-1}\right)$, which is of degree $2d$ in $Z[X]$, so that $2d \geqslant \varphi(q)$. If $q \in \{1, 2, 3, 4, 6\}$, one checks easily that $\exp\left(i2\pi\frac{p}{q}\right) \neq \frac{3+4i}{5}$. If $q = 5$ or if $q \geqslant 7$, then $\varphi(q) > 2$ so that $\cos\left(2\pi\frac{p}{q}\right)$ is not rational. Thus the root of unity μ cannot be equal to $\frac{3+4i}{5}$.

5. Some other cases of Tits' theorem

Let n be an integer with $n \geqslant 2$.

Define a subgroup Γ of $GL(n, \mathbf{C})$ [respectively of $PGL(n, \mathbf{C})$] to be *irreducible* if any linear subspace of $\mathbf{C}^n$ [resp. of $P_{\mathbf{C}}^{n-1}$] invariant by Γ is trivial, and *not almost reducible* if any subgroup of Γ of finite index is irreducible. When referring to the *Zariski* topology on $PGL(n, \mathbf{C})$, we use below the letter Z.

Reduction. Tits' theorem for complex linear groups is equivalent to the following statements (one for each $n \geqslant 2$):

Let Γ be a subgroup of $PGL(n, \mathbf{C})$ which is not almost solvable. Assume that

(i) is not almost reducible;

(ii) the Z-closure G of Γ in $PGL(n, \mathbf{C})$ is Z-connected. Then Γ contains free groups.

That one may assume (i) without loss of generality is an easy exercise on reducibility, and one may assume (ii) because the Z-closure of any subgroup of $PGL(n, \mathbf{C})$ has finitely many Z-connected components. (The hypothesis of the reduced statement are redundant: (i) and (ii) imply by Lie's theorem that G is not solvable, so that Γ is not almost solvable!)

Now let $g \in PGL(n, \mathbf{C})$ and choose a representative $\tilde{g} \in GL(n, \mathbf{C})$ of g. Let us define g to be

elliptic if $\tilde{g}$ is semi-simple with all eigenvalues of equal moduli,

parabolic if $\tilde{g}$ is not semi-simple and has all its eigenvalues of equal moduli,

hyperbolic if $\tilde{g}$ has at least two eigenvalues of distinct moduli.

These definitions are obviously independent on the choice of $\tilde{g}$. They generalize those of section 3 as follows from [Gr]. The meaning of "hyperbolic" fits with current use in dynamical systems theory (see e.g. definition 5.1 in [Sh]).

Let g be hyperbolic and let $\tilde{g}$ be as above. Let $\tilde{A}(g)$ [respectively $\tilde{A}'(g)$] be the direct sum of the nilspaces of $\tilde{g}$ corresponding to all eigenvalues of maximal modulus [resp. to all other eigenvalues] of $\tilde{g}$. Let $A(g)$ [resp. $A'(g)$] be the canonical image of $\tilde{A}(g) - \{0\}$ [resp. $\tilde{A}'(g) - \{0\}$] in $\mathbf{P} = P_{\mathbf{C}}^{n-1}$. Then $A(g) \cap A'(g) = \emptyset$ and the smallest linear subspace of $\mathbf{P}$ containing both $A(g)$ and $A'(g)$ is $\mathbf{P}$ itself. Tits calls $A(g)$ [resp. $A(g^{-1})$] the *attracting space* [resp. *repulsing space*] of g. We say that g is *sharp* if $A(g)$ is a point and that g is *very sharp* if both $A(g)$ and $A(g^{-1})$ are points. For each $k \in \{1, 2, \ldots, n-1\}$, the fundamental representation of $GL(n, C)$ in $\wedge^k \mathbf{C}^n$ induces an injection

$$\lambda_k : PGL(n, \mathbf{C}) \to PGL(\tbinom{n}{k}, \mathbf{C});$$

as g is hyperbolic, $\lambda_k(g)$ is sharp for some k. We also say that two hyperbolic elements $g, h \in PGL(n, \mathbf{C})$ are in *general position* if

$$A(g) \cup A(g^{-1}) \subset \mathbf{P} - \{A'(h) \cup A'(h^{-1})\}$$
$$A(h) \cup A(h^{-1}) \subset \mathbf{P} - \{A'(g) \cup A'(g^{-1})\}.$$

Observe that any hyperbolic element of $PGL(2, \mathbf{C})$ is very sharp, and that two hyperbolic elements of $PGL(2, \mathbf{C})$ are in general position if and only if they do not have any common fixed point on $\mathbf{S}^2$.

Recall that an element of $PGL(n, \mathbf{C})$ is *semi-simple* if its inverse image in $GL(n, \mathbf{C})$ contains diagonalisable matrices.

LEMMA 1. *Let Γ be an irreducible subgroup of $PGL(n, \mathbf{C})$ having a Z-connected Z-closure. If Γ contains a sharp semi-simple element g, then Γ contains a very sharp element.*

About the proof. Let $\tilde{g} \in GL(n, \mathbf{C})$ be some representative of g having an eigenvalue of "large" modulus and all other eigenvalues with moduli "near" 1. For suitable $h, u \in \Gamma$ and for $j \in N$ large enough, one may hope that $g^{-j}hg^jh^{-1}u$ has a representative in $GL(n, \mathbf{C})$ with one eigenvalue of very large modulus (look at $hg^jh^{-1}u$), one eigenvalue of very small modulus (look at g^{-j}), and other eigenvalues of moduli "near" 1. Section 3 of [T] shows that this hope is realistic. (See also below, after the theorem.) □

LEMMA 2. *Let Γ be an irreducible subgroup of $PGL(n, \mathbf{C})$ having a Z-connected Z-closure. If Γ contains a very sharp element, then Γ contains two very sharp elements in general position.*

Proof. Let P_1, P_2 be two linear subspaces of $\mathbf{P}$ with $P_1 \neq \emptyset$ and $P_2 \neq \mathbf{P}$. Then $\{ x \in G \mid x(P_1) \not\subset P_2 \}$ is obviously a Z-open subset of G. It is not empty:

Choose indeed $p \in P_1$; then the subspace of $\mathbf{P}$ spanned by the orbit Gp is stable under G and must therefore coincide with $\mathbf{P}$; hence there exists $x \in G$ with $x(p) \notin P_2$ and, a fortiori, $x(P_1) \not\subset P_2$.

Let g be a very sharp element in Γ. It follows from above that

$$X = \left\{ x \in G \;\middle|\; \begin{array}{l} A(g) \text{ and } A(g^{-1}) \text{ are not contained in any of } xA'(g), \\ xA'(g^{-1}), x^{-1}A'(g), x^{-1}A'(g^{-1}) \end{array} \right\}$$

is a non empty Z-open subset of G. Let $y \in X \cap \Gamma$. Then g and ygy^{-1} are both very sharp and are in general position. □

For the next lemma, we choose as above k with $1 \leqslant k \leqslant n-1$ and we consider the k^{th} fundamental representation $\lambda_k : SL(n, \mathbf{C}) \to SL(\binom{n}{k}, \mathbf{C})$ of $SL(n, \mathbf{C})$.

LEMMA. *Let* Γ *be a group and let* $\rho : \Gamma \to SL(n, \mathbf{C})$ *be an irreducible representation. Then the Z-closure* G *of* $\rho(\Gamma)$ *in* $SL(n, \mathbf{C})$ *is semi-simple and the representation* $\sigma = \lambda_k \rho : \Gamma \to SL(\binom{n}{k}, \mathbf{C})$ *is completely reducible.*

Proof. We show first that G is semi-simple. Consider the solvable radical R of G. By Lie's theorem, there exists an eigenvector for R, namely there exist $v \in \mathbf{C}^n - \{0\}$ and $\alpha \in \mathrm{Hom}(R, \mathbf{C}^*)$ with $r(v) = \alpha(r)v$ for all $r \in R$. As R is normal in G, any vector $g(v)$ $(g \in G)$ is also an eigenvector for R. By irreductibility, any vector in $\mathbf{C}^n$ is also an eigenvector, so that R is made up of dilations. But R is connected and is in $SL(n, \mathbf{C})$, so that $R = 1$.

Now $\lambda_k : G \to SL(\binom{n}{k}, \mathbf{C})$ is completely reducible; denote by $\lambda_{k,j} : G \to SL(W_j)$ the components of a decomposition $\lambda_k = \bigoplus_{j \in J} \lambda_{k,j}$ and define $\sigma_j = \lambda_{k,j}\rho$ $(j \in J)$. One has clearly $\sigma = \bigoplus_{j \in J} \sigma_j$, and each $\sigma_j : \Gamma \to SL(W_j)$ is irreducible (this because $\lambda_{k,j}$ is irreducible and by Schur's lemma). □

THEOREM. *Let* Γ *be a subgroup of* $PGL(n, \mathbf{C})$ *and assume*

(i) Γ *is neither almost solvable nor almost reducible,*

(ii) Γ *contains a semi-simple hyperbolic element.*

Then Γ *contains free groups.*

Proof. As one may consider instead of Γ a subgroup of finite index, there is no loss of generality if we assume that the Z-closure of Γ is Z-connected. We denote by $\tilde{\Gamma}$ the inverse image of Γ in $SL(n, \mathbf{C})$. By (ii), there exists $k \in \{1, \ldots, n-1\}$ and a semi-simple element $\tilde{\gamma} \in \tilde{\Gamma}$ having eigenvalues $\mu_1, \ldots, \mu_n$ with $|\mu_1| = \ldots = |\mu_k| > |\mu_j|$ for $j = k+1, \ldots, n$. Let $N = \binom{n}{k}$, and denote by λ_k both the fundamental representation $GL(n, \mathbf{C}) \to GL(N, \mathbf{C})$ and the induced homomorphism $PGL(n, \mathbf{C}) \to PGL(N, \mathbf{C})$. Then $\lambda_k(\tilde{\gamma})$ has eigenvalues $\nu_1, \ldots, \nu_N$ with $|\nu_1| > |\nu_j|$ for $j = 2, \ldots, N$. By lemma 3, there exists a $\lambda_k(\tilde{\Gamma})$-irreducible subspace W_0 of $\mathbf{C}^N$, associated to a representation $\sigma_0 : \tilde{\Gamma} \to GL(W_0)$, such that ν_1 is an eigenvalue of $\sigma_0(\tilde{\gamma})$. As the Z-closure $\tilde{G}$ of $\tilde{\Gamma}$ in $SL(n, \mathbf{C})$ is semi-simple, the group $\tilde{G}$ is perfect and $\sigma_0(\tilde{\Gamma})$ lies in $SL(W_0)$. As $|\nu_1| > 1$, one has $\dim_{\mathbf{C}} W_0 \geqslant 2$.

Thus one may assume from the start that Γ contains a sharp semi-simple element, and indeed by lemmas 1 and 2 two very sharp elements in general position. The conclusion follows as in case 2 of the proof of the proposition in section 4. □

Now lemma 1 remains true without the hypothesis "semi-simple". This has been announced by Y. Guivarch', who uses ideas of H. Fürstenberg to show the following: given an appropriate subset S of Γ containing a sharp element, then almost any "long" word in the letters of S is very sharp. Using this, one may replace (ii) in the theorem above by the following a priori weaker hypothesis

(ii′) Γ is not relatively compact.

Then, one first checks as for theorem 2 of section 4 that Γ contains hyperbolic elements; one concludes as in the previous proof, with Guivarch's version of lemma 1.

For subgroups of $PU(n)$, one may repeat the discussion at the end of section 4.

REFERENCES

[A] Ahlfors, L. V. *Möbius transformations in several dimensions.* School of Mathematics, University of Minnesota, 1981.

[Ba] Bass, H. Groups of integral representation type. *Pacific J. Math. 86* (1980), 15-51.

[BL] Bass, H. and A. Lubotzky. Automorphisms of groups and of schemes of finite type. *Preprint.*

[B] Bourbaki, N. *Eléments d'histoire des mathématiques.* Hermann 1969.

[CL] Coddington, E. A. and N. Levinson. *Theory of ordinary differential equations.* McGraw Hill, 1955.

[CG] Conze, J. P. and Y. Guivarch'. Remarques sur la distalité dans les espaces vectoriels. *C. R. Acad. Sc. Paris, Sér. A, 278* (1974), 1083-1086.

[CR] Curtis, C. and I. Reiner. *Representation theory of finite groups and associative algebras.* Interscience, 1962.

[DE] Dubins, L. E. and M. Emery. Le paradoxe de Hausdorff-Banach-Tarski. *Gazette des Mathématiciens 12* (1979), 71-76.

[D] Dixon, J. D. Free subgroups of linear groups. *Lecture Notes in Math. 319* (Springer, 1973), 45-56.

[E] Epstein, D. B. A. Almost all subgroups of a Lie group are free. *J. of Algebra 19* (1971), 261-262.

[FK] Fricke, R. and F. Klein. *Vorlesungen über die Theorie der automorphen Functionen*, vol. I. Teubner, 1897.