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also elliptic, the foot of the perpendicular from the fixed point of g onto the invariant line of g would be fixed by g, and this cannot be. If g was at the same time elliptic with fixed point  $a \in H^{n+1}$  and parabolic with fixed point  $b \in S^n$ , the line from a towards b would have two points at infinity b and b' both fixed by g, and this cannot be.

That any  $g \in GM(n)_0$  belongs to one of the three classes follows for example from Brouwer's fixed point theorem. (See also 4.9.3 in [Th].)

Observe that an hyperbolic isometry  $g \in GM(n)_0$  has a unique invariant line  $\delta$ . Suppose indeed that there are two of them, say  $\delta$  and  $\delta'$ . If  $\delta \cap \delta' \neq \phi$ , the intersection point (which is unique) is fixed by g, and this cannot be. If  $\delta \cap \delta' = \phi$  and if  $\delta$ ,  $\delta'$  have no common point at infinity, there is a unique line perpendicular to both  $\delta$  and  $\delta'$ ; but this line intersects  $\delta$  in a point fixed by g, and this cannot be. Assume finally that  $\delta \cap \delta' = \phi$  and that  $\delta$  and  $\delta'$  have a common point at infinity; choose some number  $\rho > 0$  and consider the set  $C_{\rho}$  of points in  $H^{n+1}$  at a distance of  $\rho$  from  $\delta'$ ; the intersection  $C_{\rho} \cap \delta$  is a point fixed by g, and again this cannot be. One may consequently also define an isometry  $g \in GM(n)_0$  to be

elliptic if d(a, g(a)) = 0 for some  $a \in H^{n+1}$ , parabolic if  $\inf d(a, g(a)) = 0$ , with the infimum over  $a \in H^{n+1}$  not attained, hyperbolic if  $\inf d(a, g(a)) > 0$  (and the infimum is then attained exactly on the invariant line of g).

We shall need below the following dynamical description. An hyperbolic isometry  $g \in GM(n)_0$  has on  $S^n$  one attracting point  $P_a$  and one repulsing point  $P_r$ . This means that, for any neighborhood U of  $P_a$  in  $S^n$  and for any compact subset K of  $S^n - \{P_r\}$ , one has  $g^k(K) \subset U$  for k large enough. (And similarly with  $g^{-1}$  instead of g when exchanging  $P_a$  and  $P_r$ .) Consider now a parabolic isometry  $g \in GM(n)_0$  with fixed point  $P \in S^n$ . Let U be a neighborhood of P in  $S^n$  and let K be compact in  $S^n - \{P\}$ ; then  $g^k(K) \subset U$  for any  $k \in \mathbb{Z}$  with |k| large enough. (This is obvious when g is a translation in  $\mathbb{R}^n \times \mathbb{R}_+^*$  by some vector in  $\mathbb{R}^n$ , and any parabolic isometry of  $H^{n+1}$  is conjugate to such a translation.)

# 4. Free subgroups of $GL(2, \mathbf{R})$ and of $GL(2, \mathbf{C})$

We show in this section that a subgroup of the proper Mobius group  $G = PGL(2, \mathbf{R})$  which is not almost solvable contains free groups; the same fact for  $GL(2, \mathbf{R})$  follows straightforwardly. We discuss also the case of  $GL(2, \mathbf{C})$ .

PROPOSITION. Let  $g, h \in G - \{1\}$  be without any common fixed point in  $H^2 \cup S^1$ . Then the group  $\Gamma$  generated by g and h contains free groups, up to two exceptions.

The first of these happens when  $g^2 = h^2 = 1$ . The second when one element is an involution, say  $g^2 = 1$ , when h is hyperbolic, and when g exchanges the two fixed points of h on  $S^1$ . In these two cases,  $\Gamma$  is the infinite dihedral group, and is thus solvable.

*Proof.* We check below in each of the non exceptional cases that  $\Gamma$  contains a free group.

Case 1. One element, say g, is parabolic with fixed point  $P \in S^1$ .

Consider the parabolic  $k = hgh^{-1}$ , with fixed point  $Q = h(P) \neq P$  in  $S^1$ . Let  $S_1$  [respectively  $S_2$ ] be a compact neighborhood of P [resp. Q] in  $S^1$  with  $S_1 \cap S_2 = \emptyset$ . The end of section 3 shows that there exists a positive integer  $n_0$  such that  $g^n(S_2) \subset S_1$  and  $k^n(S_1) \subset S_2$  for any  $n \in \mathbb{Z}$  with  $|n| \geq n_0$ . It follows from Klein's criterium that  $g^{n_0}$  and  $g^{n_0}$  generate a free subgroup of G.

Case 2. Both g and h are hyperbolic.

Let  $S_1$  [respectively  $S_2$ ] be a compact neighborhood of the fixed points of g [resp. of h] in  $S^1$  with  $S_1 \cap S_2 = \phi$ , and proceed as in case 1.

Case 3. One of the elements, say h, is hyperbolic with fixed points  $P, Q \in \mathbf{S}^1$  and g does not exchange them, say  $R = g(Q) \notin \{P, Q\}$ .

If  $g(P) \in \{P, Q\}$  then h and  $ghg^{-1}$  are as in case 2. We may thus assume that g(P) = Q. If  $g(R) \neq P$  then h and  $g^2hg^{-2}$  are again as in case 2. We may thus also assume g(R) = P. Consider then  $h' = g^{-1}hg$ , an hyperbolic with fixed points R and R, as well as  $h'' = ghg^{-1}hgh^{-1}g^{-1}$ , an hyperbolic with fixed points R and R are R and R are as in case 2.

Case 4. Both g and h are elliptic with  $g^2 \neq 1$ .

Possibly after conjugation within G, one may assume that  $g = r_{\alpha}$  is a rotation around the origin of the disc  $H^2$  by some angle  $\alpha \in ]0, 2\pi[-\{\pi\}]$ . Then  $k = hgh^{-1} \neq g$ , otherwise h would also fix the origin.

In the average, any point of  $S^1$  is rotated by k of an angle  $\alpha$ . More precisely, if  $\tilde{k}: \mathbf{R} \to \mathbf{R}$  is the lifting of k to the universal covering of  $S^1$  with  $0 \leq \tilde{k}(0) < 1$ ,

then  $\lim_{n\to\infty}\frac{1}{n}(\tilde{k}^n(x)-x)$  exists for all  $x\in\mathbf{R}$  and this limit is  $\alpha$ . Moreover

$$\min_{x \in \mathbb{R}} (\tilde{k}(x) - x) \leq \alpha \leq \max_{x \in \mathbb{R}} (\tilde{k}(x) - x).$$

(See any exposition of the rotation number, for example chapter 17 in [CL] or section 1 in [Ka].) It follows that there exists  $P \in S^1$  with k(P) = g(P), so that  $g^{-1}k$  has a fixed point in  $S^1$  and one of the previous cases applies.

Exceptional cases. If  $g^2 = h^2 = 1$ , then gh generate an infinite cyclic subgroup of index 2 in  $\Gamma$  and  $\Gamma$  is isomorphic to the infinite dihedral group. If h is hyperbolic and if g exchanges its fixed points, then  $ghg^{-1} = h^{-1}$  so that  $g^2 = (gh)^2 = 1$  and  $\Gamma$  is as in the previous case.

The proof is now complete.

The proposition above is well known, and may essentially be found in any of the following papers: [LU1], [Md], [Ro] (see corollary 1). One should also mention Magnus' surveys [Ms1], [Ms2].

As two elements of G having a common fixed point in  $H^2 \cup S^1$  generate a solvable subgroup, we have proved the 2-generators particular case of the following fact.

Theorem 1. A subgroup  $\Gamma$  of  $G = PGL(2, \mathbf{R})$  (or of  $GL(2, \mathbf{R})$ ) which is not solvable contains free groups.

*Proof.* We assume that  $\Gamma$  does not contain free groups, and check that  $\Gamma$  is solvable. If  $\Gamma$  contains at least one parabolic isometry, this follows from case 1 of the proof above. If it contains at least one hyperbolic isometry, then all hyperbolics in  $\Gamma$  have a common fixed point (see case 2) and then either all elements in  $\Gamma$  have a common fixed point or  $\Gamma$  is dihedral (see case 3). Finally, if  $\Gamma$  is an elliptic group, it follows from case 4 that  $\Gamma$  is abelian.

This covers in particular the case of Fuchsian groups. The next theorem covers that of Kleinian groups.

Theorem 2. Let  $\Gamma$  be a subgroup of  $SL(2, \mathbb{C})$  which is not solvable. Assume moreover that  $\Gamma$  is not relatively compact (or equivalently that  $\Gamma$  is not conjugate to a subgroup of the maximal compact subgroup SU(2) of  $SL(2, \mathbb{C})$ ). Then  $\Gamma$  contains free groups.

In particular, a discrete subgroup of  $PGL(2, \mathbb{C})$  which is not almost solvable contains free groups.

*Proof.* The group  $\Gamma$  acts on  $\mathbb{C}^2$ ; as  $\Gamma$  is not solvable, the representation is irreducible. Easy arguments à la Burnside show that  $\Gamma$  does not contain elliptic elements only; indeed,  $\Gamma$  does contain a hyperbolic element (see [CG], or corollary 1.8 in [B]). The first statement follows now as theorem 1.

The second follows from this: a discrete subgroup of  $PGL(2, \mathbb{C})$  containing elliptic elements only is finite. Indeed, such a group is periodic. If  $\Gamma$  is a priori

known to be finitely generated, then  $\Gamma$  is finite by a theorem of Schur (§36 in [CR]) so that the hyperbolic subspace  $F(\Gamma) = \{ x \in H^3 \mid \Gamma x = \{x\} \}$  is non empty. In general, to any finitely generated subgroup  $\Gamma_1$  of  $\Gamma$  corresponds a non empty subspace  $F_1 \subset H^n$ ; it is easy to check that  $F(\Gamma) = \bigcap F_1$  is non empty so that  $\Gamma$  lies in a compact subgroup of the Mœbius group; it follows again that  $\Gamma$  is finite.

Instead of the assumption of theorem 2, assume the following: there exists  $g \in \Gamma$  with two distinct eigenvalues of same modulus, say  $\mu_1 = \rho \exp{(i\theta_1)}$  and  $\mu_2 = \rho \exp{(i\theta_2)}$  where  $\rho$ ,  $\theta_1$ ,  $\theta_2 \in \mathbf{R}$  satisfy  $\rho > 0$  and  $\theta_1 \not\equiv \theta_2 \pmod{2\pi}$ , and there exists an automorphism  $\alpha$  of  $\mathbf{C}$  with  $|\alpha(\mu_1)| \neq |\alpha(\mu_2)|$ . Then  $\alpha$  induces an automorphism  $\tilde{\alpha}$  of  $GL(2, \mathbf{C})$  and the proof applies to  $\tilde{\alpha}(\Gamma)$ . But this procedure has its limits, because there exist complex numbers  $\mu$  (such as  $\frac{1}{5}(3+4i)$ , see the remark below) such that  $|\alpha(\mu)| = 1$  for any automorphism  $\alpha$  of  $\mathbf{C}$  but which are not roots of 1; then the argument above fails 1) for example for  $g = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$ .

Something is true however: let k be a finitely generated field of characteristic 0, let  $\mu \in k - \{0\}$  and assume  $\mu$  is not a root of 1. Then there exists a locally compact field k' endowed with an absolute value  $\omega$  and there exists a homomorphism  $\sigma: k \to k'$  such that  $\omega(\sigma(\mu)) \neq 1$ ; this is lemma 4.1 of [T]. It follows that the argument above may be recuperated, but one has to consider other fields than subfields of  $\mathbb{C}$ .

For self-consistency, let us end with the announced remark. For any automorphism  $\alpha$  of C, one has clearly

$$\left| \alpha \left( \frac{3+4i}{5} \right) \right| = \left| \frac{3\pm 4i}{5} \right| = 1;$$

we check now that  $\frac{3+4i}{5}$  is not a root of one.

Let p, q be coprime integers and let  $\mu = \exp\left(i2\pi\frac{p}{q}\right)$  be a root of 1. Then  $\mu$  is an algebraic number of degree  $\phi(q)$ , where  $\phi$  is Euler's function. It follows that  $\cos\left(2\pi\frac{p}{q}\right)$  is an algebraic number of degree  $d \geqslant \frac{1}{2}\phi(q)$ : because if F is a polynomial of degree d in Z[X] with  $F\left(\cos\left(2\pi\frac{p}{q}\right)\right) = 0$ , then  $\mu$  is a root of

<sup>1)</sup> This shows that one point on page 50 of [D] is incorrect.

 $X^dF\left(\frac{1}{2}X+\frac{1}{2}X^{-1}\right)$ , which is of degree 2d in Z[X], so that  $2d\geqslant \varphi(q)$ . If  $q\in\{1,2,3,4,6\}$ , one checks easily that  $\exp\left(i2\pi\frac{p}{q}\right)\neq\frac{3+4i}{5}$ . If q=5 or if  $q\geqslant 7$ , then  $\varphi(q)>2$  so that  $\cos\left(2\pi\frac{p}{q}\right)$  is not rational. Thus the root of unity  $\mu$  cannot be equal to  $\frac{3+4i}{5}$ .

# 5. Some other cases of Tits' theorem

Let *n* be an integer with  $n \ge 2$ .

Define a subgroup  $\Gamma$  of  $GL(n, \mathbb{C})$  [respectively of  $PGL(n, \mathbb{C})$ ] to be *irreducible* if any linear subspace of  $\mathbb{C}^n$  [resp. of  $P_{\mathbb{C}}^{n-1}$ ] invariant by  $\Gamma$  is trivial, and *not almost* reducible if any subgroup of  $\Gamma$  of finite index is irreducible. When referring to the Zariski topology on  $PGL(n, \mathbb{C})$ , we use below the letter  $\mathbb{Z}$ .

Reduction. Tits' theorem for complex linear groups is equivalent to the following statements (one for each  $n \ge 2$ ):

Let  $\Gamma$  be a subgroup of  $PGL(n, \mathbb{C})$  which is not almost solvable. Assume that

- (i) is not almost reducible;
- (ii) the Z-closure G of  $\Gamma$  in  $PGL(n, \mathbb{C})$  is Z-connected. Then  $\Gamma$  contains free groups.

That one may assume (i) without loss of generality is an easy exercise on reducibility, and one may assume (ii) because the Z-closure of any subgroup of  $PGL(n, \mathbb{C})$  has finitely many Z-connected components. (The hypothesis of the reduced statement are redundant: (i) and (ii) imply by Lie's theorem that G is not solvable, so that  $\Gamma$  is not almost solvable!)

Now let  $g \in PGL(n, \mathbb{C})$  and choose a representative  $\tilde{g} \in GL(n, \mathbb{C})$  of g. Let us define g to be

elliptic if  $\tilde{g}$  is semi-simple with all eigenvalues of equal moduli, parabolic if  $\tilde{g}$  is not semi-simple and has all its eigenvalues of equal moduli, hyperbolic if  $\tilde{g}$  has at least two eigenvalues of distinct moduli.

These definitions are obviously independent on the choice of  $\tilde{g}$ . They generalize those of section 3 as follows from [Gr]. The meaning of "hyperbolic" fits with current use in dynamical systems theory (see e.g. definition 5.1 in [Sh]).

Let g be hyperbolic and let  $\tilde{g}$  be as above. Let  $\tilde{A}(g)$  [respectively  $\tilde{A}'(g)$ ] be the direct sum of the nilspaces of  $\tilde{g}$  corresponding to all eigenvalues of maximal modulus [resp. to all other eigenvalues] of  $\tilde{g}$ . Let A(g) [resp. A'(g)] be the canonical image of  $\tilde{A}(g) - \{0\}$  [resp.  $\tilde{A}'(g) - \{0\}$ ] in  $\mathbf{P} = P_{\mathbf{C}}^{n-1}$ . Then  $A(g) \cap A'(g) = \emptyset$  and the smallest linear subspace of  $\mathbf{P}$  containing both A(g) and A'(g) is  $\mathbf{P}$  itself. Tits calls A(g) [resp.  $A(g^{-1})$ ] the attracting space [resp. repulsing space] of g. We say that g is sharp if A(g) is a point and that g is very sharp if both A(g) and  $A(g^{-1})$  are points. For each  $k \in \{1, 2, ..., n-1\}$ , the fundamental representation of GL(n, C) in  $\wedge^k \mathbf{C}^n$  induces an injection

$$\lambda_k: PGL(n, \mathbb{C}) \to PGL(\binom{n}{k}, \mathbb{C});$$

as g is hyperbolic,  $\lambda_k(g)$  is sharp for some k. We also say that two hyperbolic elements  $g, h \in PGL(n, \mathbb{C})$  are in general position if

$$A(g) \cup A(g^{-1}) \subset \mathbf{P} - \{A'(h) \cup A'(h^{-1})\}\$$
  
 $A(h) \cup A(h^{-1}) \subset \mathbf{P} - \{A'(g) \cup A'(g^{-1})\}\$ .

Observe that any hyperbolic element of  $PGL(2, \mathbb{C})$  is very sharp, and that two hyperbolic elements of  $PGL(2, \mathbb{C})$  are in general position if and only if they do not have any common fixed point on  $\mathbb{S}^2$ .

Recall that an element of  $PGL(n, \mathbb{C})$  is semi-simple if its inverse image in  $GL(n, \mathbb{C})$  contains diagonalisable matrices.

LEMMA 1. Let  $\Gamma$  be an irreducible subgroup of  $PGL(n, \mathbb{C})$  having a Z-connected Z-closure. If  $\Gamma$  contains a sharp semi-simple element g, then  $\Gamma$  contains a very sharp element.

About the proof. Let  $\tilde{g} \in GL(n, \mathbb{C})$  be some representative of g having an eigenvalue of "large" modulus and all other eigenvalues with moduli "near" 1. For suitable  $h, u \in \Gamma$  and for  $j \in N$  large enough, one may hope that  $g^{-j}hg^{j}h^{-1}u$  has a representative in  $GL(n, \mathbb{C})$  with one eigenvalue of very large modulus (look at  $hg^{j}h^{-1}u$ ), one eigenvalue of very small modulus (look at  $g^{-j}$ ), and other eigenvalues of moduli "near" 1. Section 3 of [T] shows that this hope is realistic. (See also below, after the theorem.)

LEMMA 2. Let  $\Gamma$  be an irreducible subgroup of  $PGL(n, \mathbb{C})$  having a Z-connected Z-closure. If  $\Gamma$  contains a very sharp element, then  $\Gamma$  contains two very sharp elements in general position.

*Proof.* Let  $P_1$ ,  $P_2$  be two linear subspaces of **P** with  $P_1 \neq \emptyset$  and  $P_2 \neq \mathbf{P}$ . Then  $\{x \in G \mid x(P_1) \neq P_2\}$  is obviously a Z-open subset of G. It is not empty:

Choose indeed  $p \in P_1$ ; then the subspace of **P** spanned by the orbit Gp is stable under G and must therefore coincide with **P**; hence there exists  $x \in G$  with  $x(p) \notin P_2$  and, a fortiori,  $x(P_1) \notin P_2$ .

Let g be a very sharp element in  $\Gamma$ . It follows from above that

$$X = \left\{ x \in G \middle| \begin{array}{l} A(g) \text{ and } A(g^{-1}) \text{ are not contained in any of } xA'(g), \\ xA'(g^{-1}), x^{-1}A'(g), x^{-1}A'(g^{-1}) \end{array} \right\}$$

is a non empty Z-open subset of G. Let  $y \in X \cap \Gamma$ . Then g and  $ygy^{-1}$  are both very sharp and are in general position.

For the next lemma, we choose as above k with  $1 \le k \le n-1$  and we consider the  $k^{\text{th}}$  fundamental representation  $\lambda_k : SL(n, \mathbb{C}) \to SL(\binom{n}{k}, \mathbb{C})$  of  $SL(n, \mathbb{C})$ .

LEMMA. Let  $\Gamma$  be a group and let  $\rho: \Gamma \to SL(n, \mathbb{C})$  be an irreducible representation. Then the Z-closure G of  $\rho(\Gamma)$  in  $SL(n, \mathbb{C})$  is semi-simple and the representation  $\sigma = \lambda_k \rho: \Gamma \to SL(\binom{n}{k}, \mathbb{C})$  is completely reducible.

*Proof.* We show first that G is semi-simple. Consider the solvable radical R of G. By Lie's theorem, there exists an eigenvector for R, namely there exist  $v \in \mathbb{C}^n - \{0\}$  and  $\alpha \in \text{Hom}(R, \mathbb{C}^*)$  with  $r(v) = \alpha(r)v$  for all  $r \in R$ . As R is normal in G, any vector g(v) ( $g \in G$ ) is also an eigenvector for R. By irreductibility, any vector in  $\mathbb{C}^n$  is also an eigenvector, so that R is made up of dilations. But R is connected and is in  $SL(n, \mathbb{C})$ , so that R = 1.

Now  $\lambda_k: G \to SL(\binom{n}{k}, \mathbb{C})$  is completely reducible; denote by  $\lambda_{k,j}: G \to SL(W_j)$  the components of a decomposition  $\lambda_k = \bigoplus_{j \in J} \lambda_{k,j}$  and define  $\sigma_j = \lambda_{k,j} \rho$  ( $j \in J$ ). One has clearly  $\sigma = \bigoplus_{j \in J} \sigma_j$ , and each  $\sigma_j: \Gamma \to SL(W_j)$  is irreducible (this because  $\lambda_{k,j}$  is irreducible and by Schur's lemma).

Theorem. Let  $\Gamma$  be a subgroup of  $PGL(n, \mathbb{C})$  and assume

- (i)  $\Gamma$  is neither almost solvable nor almost reducible,
- (ii)  $\Gamma$  contains a semi-simple hyperbolic element.

Then  $\Gamma$  contains free groups.

*Proof.* As one may consider instead of  $\Gamma$  a subgroup of finite index, there is no loss of generality if we assume that the Z-closure of  $\Gamma$  is Z-connected. We denote by  $\widetilde{\Gamma}$  the inverse image of  $\Gamma$  in  $SL(n, \mathbb{C})$ . By (ii), there exists  $k \in \{1, ..., n-1\}$  and a semi-simple element  $\widetilde{\gamma} \in \widetilde{\Gamma}$  having eigenvalues  $\mu_1, ..., \mu_n$  with  $|\mu_1| = ...$   $= |\mu_k| > |\mu_j|$  for j = k + 1, ..., n. Let  $N = \binom{n}{k}$ , and denote by  $\lambda_k$  both the fundamental representation  $GL(n, \mathbb{C}) \to GL(N, \mathbb{C})$  and the induced

homomorphism  $PGL(n, \mathbb{C}) \to PGL(N, \mathbb{C})$ . Then  $\lambda_k(\tilde{\gamma})$  has eigenvalues  $v_1, ..., v_N$  with  $|v_1| > |v_j|$  for j = 2, ..., N. By lemma 3, there exists a  $\lambda_k(\tilde{\Gamma})$ -irreducible subspace  $W_0$  of  $\mathbb{C}^N$ , associated to a representation  $\sigma_0 \colon \tilde{\Gamma} \to GL(W_0)$ , such that  $v_1$  is an eigenvalue of  $\sigma_0(\tilde{\gamma})$ . As the Z-closure  $\tilde{G}$  of  $\tilde{\Gamma}$  in  $SL(n, \mathbb{C})$  is semi-simple, the group  $\tilde{G}$  is perfect and  $\sigma_0(\tilde{\Gamma})$  lies in  $SL(W_0)$ . As  $|v_1| > 1$ , one has  $\dim_{\mathbb{C}} W_0 \ge 2$ .

Thus one may assume from the start that  $\Gamma$  contains a sharp semi-simple element, and indeed by lemmas 1 and 2 two very sharp elements in general position. The conclusion follows as in case 2 of the proof of the proposition in section 4.

Now lemma 1 remains true without the hypothesis "semi-simple". This has been announced by Y. Guivarch', who uses ideas of H. Fürstenberg to show the following: given an appropriate subset S of  $\Gamma$  containing a sharp element, then almost any "long" word in the letters of S is very sharp. Using this, one may replace (ii) in the theorem above by the following a priori weaker hypothesis

(ii')  $\Gamma$  is not relatively compact.

Then, one first checks as for theorem 2 of section 4 that  $\Gamma$  contains hyperbolic elements; one concludes as in the previous proof, with Guivarch's version of lemma 1.

For subgroups of PU(n), one may repeat the discussion at the end of section 4.

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