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proof requires Tits' theorem is due to Gromov: a finitely generated group has polynomial growth if and only if it is almost nilpotent [G].

The analogue of Tits' theorem for division rings does not hold as such [L1], but conjectural statements have been formulated [L2]. Another generalisation of the theorem is proposed as a research problem in remark 1.4.2 of [BL].

3. DIGRESSION ON HYPERBOLIC GEOMETRY

Let n be an integer, $n \geq 1$. The hyperbolic space H^{n+1} of dimension $n + 1$ is the open unit ball of the euclidean space \mathbf{R}^{n+1} . Hyperbolic lines (called lines below) in H^{n+1} are traces on H^{n+1} of circles and euclidean lines in \mathbf{R}^{n+1} which are orthogonal to \mathbf{S}^n . Two distinct points $P, Q \in H^{n+1}$ are on a unique line which determines two points $P_\infty, Q_\infty \in \mathbf{S}^n$, say with P, Q, Q_∞, P_∞ arranged in cyclic order on the euclidean circle defining this line. The (hyperbolic) distance between P and Q is given by a cross-ratio of euclidean distances; more precisely, it is defined to be

$$d(P, Q) = \text{Log}(P, Q, Q_\infty, P_\infty) = \log \left(\frac{|P - Q_\infty|}{|P - P_\infty|} : \frac{|Q - Q_\infty|}{|Q - P_\infty|} \right).$$

The *proper Mæbius group* $GM(n)_0$ is the group of orientation preserving isometries of \mathbf{R}^{n+1} for this distance. Any $g \in GM(n)_0$ extends to a homeomorphism of the closed ball $H^{n+1} \cup \mathbf{S}^n$. One may check that $GM(1)_0$ is isomorphic to $PGL(2, \mathbf{R})$ and $GM(2)_0$ to $PGL(2, \mathbf{C})$.

There is an equivalent description with H^{n+1} the half space $\mathbf{R}^n \times \mathbf{R}_+^*$. The set of "points at infinity" is then $\mathbf{R}^n \cup \{\infty\}$ rather than \mathbf{S}^n .

For all this, see e.g. [A] or [Si].

An isometry $g \in GM(n)_0$ is said to be

elliptic if there is some point in H^{n+1} fixed by g ,

parabolic if there is in \mathbf{S}^n exactly one point fixed by g ,

hyperbolic if there is a line in H^{n+1} invariant by g on which g has no fixed point.

(Following Thurston [Th], we call "hyperbolic" elements which are "loxodromic" in classical literature, such as in [Gr].)

PROPOSITION. *Elliptic, parabolic and hyperbolic elements define a partition of the proper Mæbius group in three disjoint classes.*

Proof. Let us first check that the three classes do not overlap in $GM(n)_0$. If g is hyperbolic, it has two fixed points in \mathbf{S}^n and thus cannot be parabolic; if g was

also elliptic, the foot of the perpendicular from the fixed point of g onto the invariant line of g would be fixed by g , and this cannot be. If g was at the same time elliptic with fixed point $a \in H^{n+1}$ and parabolic with fixed point $b \in S^n$, the line from a towards b would have two points at infinity b and b' both fixed by g , and this cannot be.

That any $g \in GM(n)_0$ belongs to one of the three classes follows for example from Brouwer's fixed point theorem. (See also 4.9.3 in [Th].) \square

Observe that an hyperbolic isometry $g \in GM(n)_0$ has a unique invariant line δ . Suppose indeed that there are two of them, say δ and δ' . If $\delta \cap \delta' \neq \phi$, the intersection point (which is unique) is fixed by g , and this cannot be. If $\delta \cap \delta' = \phi$ and if δ, δ' have no common point at infinity, there is a unique line perpendicular to both δ and δ' ; but this line intersects δ in a point fixed by g , and this cannot be. Assume finally that $\delta \cap \delta' = \phi$ and that δ and δ' have a common point at infinity; choose some number $\rho > 0$ and consider the set C_ρ of points in H^{n+1} at a distance of ρ from δ' ; the intersection $C_\rho \cap \delta$ is a point fixed by g , and again this cannot be. One may consequently also define an isometry $g \in GM(n)_0$ to be

elliptic if $d(a, g(a)) = 0$ for some $a \in H^{n+1}$,

parabolic if $\inf d(a, g(a)) = 0$, with the infimum over $a \in H^{n+1}$ not attained,

hyperbolic if $\inf d(a, g(a)) > 0$ (and the infimum is then attained exactly on the invariant line of g).

We shall need below the following *dynamical description*. An hyperbolic isometry $g \in GM(n)_0$ has on S^n one attracting point P_a and one repulsing point P_r . This means that, for any neighborhood U of P_a in S^n and for any compact subset K of $S^n - \{P_r\}$, one has $g^k(K) \subset U$ for k large enough. (And similarly with g^{-1} instead of g when exchanging P_a and P_r .) Consider now a parabolic isometry $g \in GM(n)_0$ with fixed point $P \in S^n$. Let U be a neighborhood of P in S^n and let K be compact in $S^n - \{P\}$; then $g^k(K) \subset U$ for any $k \in \mathbf{Z}$ with $|k|$ large enough. (This is obvious when g is a translation in $\mathbf{R}^n \times \mathbf{R}_+^*$ by some vector in \mathbf{R}^n , and any parabolic isometry of H^{n+1} is conjugate to such a translation.)

4. FREE SUBGROUPS OF $GL(2, \mathbf{R})$ AND OF $GL(2, \mathbf{C})$

We show in this section that a subgroup of the proper Mœbius group $G = PGL(2, \mathbf{R})$ which is not almost solvable contains free groups; the same fact for $GL(2, \mathbf{R})$ follows straightforwardly. We discuss also the case of $GL(2, \mathbf{C})$.