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proof requires Tits' theorem is due to Gromov: a finitely generated group has polynomial growth if and only if it is almost nilpotent [G].

The analogue of Tits' theorem for division rings does not hold as such [L1], but conjectural statements have been formulated [L2]. Another generalisation of the theorem is proposed as a research problem in remark 1.4.2 of [BL].

### 3. DIGRESSION ON HYPERBOLIC GEOMETRY

Let  $n$  be an integer,  $n \geq 1$ . The hyperbolic space  $H^{n+1}$  of dimension  $n + 1$  is the open unit ball of the euclidean space  $\mathbf{R}^{n+1}$ . Hyperbolic lines (called lines below) in  $H^{n+1}$  are traces on  $H^{n+1}$  of circles and euclidean lines in  $\mathbf{R}^{n+1}$  which are orthogonal to  $\mathbf{S}^n$ . Two distinct points  $P, Q \in H^{n+1}$  are on a unique line which determines two points  $P_\infty, Q_\infty \in \mathbf{S}^n$ , say with  $P, Q, Q_\infty, P_\infty$  arranged in cyclic order on the euclidean circle defining this line. The (hyperbolic) distance between  $P$  and  $Q$  is given by a cross-ratio of euclidean distances; more precisely, it is defined to be

$$d(P, Q) = \text{Log}(P, Q, Q_\infty, P_\infty) = \log \left( \frac{|P - Q_\infty|}{|P - P_\infty|} : \frac{|Q - Q_\infty|}{|Q - P_\infty|} \right).$$

The *proper Mæbius group*  $GM(n)_0$  is the group of orientation preserving isometries of  $\mathbf{R}^{n+1}$  for this distance. Any  $g \in GM(n)_0$  extends to a homeomorphism of the closed ball  $H^{n+1} \cup \mathbf{S}^n$ . One may check that  $GM(1)_0$  is isomorphic to  $PGL(2, \mathbf{R})$  and  $GM(2)_0$  to  $PGL(2, \mathbf{C})$ .

There is an equivalent description with  $H^{n+1}$  the half space  $\mathbf{R}^n \times \mathbf{R}_+^*$ . The set of "points at infinity" is then  $\mathbf{R}^n \cup \{\infty\}$  rather than  $\mathbf{S}^n$ .

For all this, see e.g. [A] or [Si].

An isometry  $g \in GM(n)_0$  is said to be

*elliptic* if there is some point in  $H^{n+1}$  fixed by  $g$ ,

*parabolic* if there is in  $\mathbf{S}^n$  exactly one point fixed by  $g$ ,

*hyperbolic* if there is a line in  $H^{n+1}$  invariant by  $g$  on which  $g$  has no fixed point.

(Following Thurston [Th], we call "hyperbolic" elements which are "loxodromic" in classical literature, such as in [Gr].)

**PROPOSITION.** *Elliptic, parabolic and hyperbolic elements define a partition of the proper Mæbius group in three disjoint classes.*

*Proof.* Let us first check that the three classes do not overlap in  $GM(n)_0$ . If  $g$  is hyperbolic, it has two fixed points in  $\mathbf{S}^n$  and thus cannot be parabolic; if  $g$  was

also elliptic, the foot of the perpendicular from the fixed point of  $g$  onto the invariant line of  $g$  would be fixed by  $g$ , and this cannot be. If  $g$  was at the same time elliptic with fixed point  $a \in H^{n+1}$  and parabolic with fixed point  $b \in S^n$ , the line from  $a$  towards  $b$  would have two points at infinity  $b$  and  $b'$  both fixed by  $g$ , and this cannot be.

That any  $g \in GM(n)_0$  belongs to one of the three classes follows for example from Brouwer's fixed point theorem. (See also 4.9.3 in [Th].)  $\square$

Observe that an hyperbolic isometry  $g \in GM(n)_0$  has a unique invariant line  $\delta$ . Suppose indeed that there are two of them, say  $\delta$  and  $\delta'$ . If  $\delta \cap \delta' \neq \emptyset$ , the intersection point (which is unique) is fixed by  $g$ , and this cannot be. If  $\delta \cap \delta' = \emptyset$  and if  $\delta, \delta'$  have no common point at infinity, there is a unique line perpendicular to both  $\delta$  and  $\delta'$ ; but this line intersects  $\delta$  in a point fixed by  $g$ , and this cannot be. Assume finally that  $\delta \cap \delta' = \emptyset$  and that  $\delta$  and  $\delta'$  have a common point at infinity; choose some number  $\rho > 0$  and consider the set  $C_\rho$  of points in  $H^{n+1}$  at a distance of  $\rho$  from  $\delta'$ ; the intersection  $C_\rho \cap \delta$  is a point fixed by  $g$ , and again this cannot be. One may consequently also define an isometry  $g \in GM(n)_0$  to be

*elliptic* if  $d(a, g(a)) = 0$  for some  $a \in H^{n+1}$ ,

*parabolic* if  $\inf d(a, g(a)) = 0$ , with the infimum over  $a \in H^{n+1}$  not attained,

*hyperbolic* if  $\inf d(a, g(a)) > 0$  (and the infimum is then attained exactly on the invariant line of  $g$ ).

We shall need below the following *dynamical description*. An hyperbolic isometry  $g \in GM(n)_0$  has on  $S^n$  one attracting point  $P_a$  and one repulsing point  $P_r$ . This means that, for any neighborhood  $U$  of  $P_a$  in  $S^n$  and for any compact subset  $K$  of  $S^n - \{P_r\}$ , one has  $g^k(K) \subset U$  for  $k$  large enough. (And similarly with  $g^{-1}$  instead of  $g$  when exchanging  $P_a$  and  $P_r$ .) Consider now a parabolic isometry  $g \in GM(n)_0$  with fixed point  $P \in S^n$ . Let  $U$  be a neighborhood of  $P$  in  $S^n$  and let  $K$  be compact in  $S^n - \{P\}$ ; then  $g^k(K) \subset U$  for any  $k \in \mathbb{Z}$  with  $|k|$  large enough. (This is obvious when  $g$  is a translation in  $\mathbb{R}^n \times \mathbb{R}_+^*$  by some vector in  $\mathbb{R}^n$ , and any parabolic isometry of  $H^{n+1}$  is conjugate to such a translation.)

#### 4. FREE SUBGROUPS OF $GL(2, \mathbb{R})$ AND OF $GL(2, \mathbb{C})$

We show in this section that a subgroup of the proper Möbius group  $G = PGL(2, \mathbb{R})$  which is not almost solvable contains free groups; the same fact for  $GL(2, \mathbb{R})$  follows straightforwardly. We discuss also the case of  $GL(2, \mathbb{C})$ .