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In 1914, this example allowed Hausdorff to show that there does not exist any finitely additive rotation-invariant measure defined on all subsets of the sphere  $S^2$ . See [H], and [DE] for subsequent history. While discussing this, let us mention the following open problem (brought to my attention by M. Keane): does there exist a finitely additive probability measure on the Borel subsets of  $S^2$ , vanishing on meagre sets, invariant under rotations? (The answer for countably additive measures is no, and follows from the unicity of Haar measure on a compact group; see e.g. §9 in [Wi].)

*Remark.* Let  $G$  be a connected real Lie group. Then  $G$  contains at least one subgroup isomorphic to the free group on two generators  $F_2$  if and only if  $G$  is not solvable, as results from standard Lie theory as follows.

To check the non trivial implication, we assume that  $G$  is not solvable, so that  $G$  contains a semi-simple subgroup  $S$  by a theorem of Levi and Mal'cev. Consider a Cartan decomposition  $\mathfrak{s} = \mathfrak{k} \oplus \mathfrak{p}$  of the Lie algebra of  $S$ . If  $\mathfrak{k} \neq \{0\}$ , root theory shows that the semi-simple compact algebra  $\mathfrak{k}$  contains a subalgebra isomorphic to  $\mathfrak{su}(2)$ , so that  $G$  contains a subgroup isomorphic to one of  $SU(2)$ ,  $SO(3)$ . If  $\mathfrak{k} = \{0\}$ , then  $\mathfrak{s}$  is split and root theory again shows that  $\mathfrak{s}$  contains a copy of  $\mathfrak{sl}(2, \mathbf{R})$ , so that  $G$  contains a subgroup isomorphic to a covering of  $PSL(2, \mathbf{R})$ . In all cases, examples above show that  $G$  contains a copy of  $F_2$ .

So, let  $G$  be a connected Lie group containing a copy of  $F_2$ . For  $w \in F_2 - \{1\}$  and  $g, h \in G$ , let  $w(g, h)$  be the element of  $G$  obtained when replacing the two generators of  $F_2$  by  $g$  and  $h$  in  $w$ . Then

$$X_w = \{ (g, h) \in G \times G \mid w(g, h) = 1 \}$$

has empty interior (think of analytic continuation). It follows from Baire's theorem that the set  $G \times G - \bigcup X_w$  (union over  $w \in F_2 - \{1\}$ ) of those  $(g, h) \in G \times G$  such that  $g$  and  $h$  generate a free group is dense and has full measure in  $G \times G$  [E]. (If  $G$  is moreover semi-simple, it follows from a note by Kuranishi and from Tits' theorem that there exist  $g, h \in G$  generating a subgroup of  $G$  which is both free and dense [Ku].)

## 2. STATEMENT OF TITS' THEOREM

Recall that, given a group  $\Gamma$ , its derived group  $D\Gamma$  is the subgroup generated by elements of the form  $ghg^{-1}h^{-1}$  and that  $\Gamma$  is *solvable* if  $D(\dots D(\Gamma) \dots) = \{1\}$  for sufficiently many  $D$ 's. We say that  $\Gamma$  is *almost solvable* (other people say virtually solvable) if it contains a solvable subgroup of finite index. For example, groups of

triangular matrices are solvable and non abelian free groups are not almost solvable. By “free group”, we mean hereafter *non abelian free group*.

A *linear group* over a field  $\mathbf{K}$  is a group which has at least one faithful finite dimensional representation over  $\mathbf{K}$ , namely a group isomorphic to a subgroup of  $GL(n, \mathbf{K})$  for some  $n$ . Groups are far from being all linear, even under the hypothesis of finite generation. Famous examples of non linear groups are the quotients  $F_2/F_2^m$  for  $m$  odd and large enough, where  $F_2^m$  is the subgroup of the free group  $F_2$  generated by elements of the form  $g^m$ . (Novikov’s negative solution to the Burnside problem; in the original paper,  $m$  large enough means  $m \geq 4381$ .)

Easier examples are provided by finitely generated infinite simple groups (there is such a group, discovered by G. Higman, which is described in [S], n° I.1.4). They are not linear, because it is a result of Mal’cev that a finitely generated linear group  $\Gamma$  is residually finite [M]. (This means that, for any  $\gamma \in \Gamma - \{1\}$ , there exists a homomorphism  $\varphi$  of  $\Gamma$  onto a finite group with  $\varphi(\gamma) \neq 1$ ; instructive and easy exercice: check that  $SL(n, \mathbf{Z})$  is residually finite.)

Also, any finitely generated non hopfian group cannot be linear ( $\Gamma$  is non hopfian if there exists a non injective homomorphism of  $\Gamma$  onto itself); an example of such a group is that generated by two elements  $g, h$  submitted to the relation  $h^{-1}g^2h = g^3$  (see [LS], page 197).

**TITS’ THEOREM.** *A linear group  $\Gamma$  over a field  $\mathbf{K}$  of characteristic 0 which is not almost solvable contains a free group.*

This theorem has been conjectured by Bass and Serre, and proved in [T] together with other results, some concerning positive characteristics.

The following precision has been added by Wang [Wa]: there exists for each positive integer  $n$  a constant  $\lambda(n)$  such that any subgroup of  $GL(n, \mathbf{K})$  without free subgroup contains a solvable subgroup of index smaller than  $\lambda(n)$ .

Let  $\Gamma$  be a group having a finite set of generators  $S$  which is a subgroup of  $GL(n, \mathbf{K})$  for some  $n$ . If  $k$  is the subfield of  $\mathbf{K}$  generated by entries of elements of  $S$ , then  $\Gamma \subset GL(n, k)$ . As  $k$  is finitely generated of characteristic zero, there exists an embedding of  $k$  in  $\mathbf{C}$  and one may assume that  $\Gamma$  lies in  $GL(n, \mathbf{C})$ . For finitely generated groups (and also in the general case by [Wh]), it is consequently sufficient to prove Tits’ theorem for  $\mathbf{K} = \mathbf{C}$  (or  $\mathbf{K} = \mathbf{R}$  because  $GL(n, \mathbf{C})$  is a subgroup of  $GL(2n, \mathbf{R})$ ). But this apparent simplification (?) is deceptive, because the proof does require other fields than fields of complex numbers.

It follows from the theorem that a linear group over a field of characteristic zero which is not amenable contains a free group; this answers for linear groups a question formulated by J. von Neumann [vN]. Another famous result whose

proof requires Tits' theorem is due to Gromov: a finitely generated group has polynomial growth if and only if it is almost nilpotent [G].

The analogue of Tits' theorem for division rings does not hold as such [L1], but conjectural statements have been formulated [L2]. Another generalisation of the theorem is proposed as a research problem in remark 1.4.2 of [BL].

### 3. DIGRESSION ON HYPERBOLIC GEOMETRY

Let  $n$  be an integer,  $n \geq 1$ . The hyperbolic space  $H^{n+1}$  of dimension  $n+1$  is the open unit ball of the euclidean space  $\mathbf{R}^{n+1}$ . Hyperbolic lines (called lines below) in  $H^{n+1}$  are traces on  $H^{n+1}$  of circles and euclidean lines in  $\mathbf{R}^{n+1}$  which are orthogonal to  $\mathbf{S}^n$ . Two distinct points  $P, Q \in H^{n+1}$  are on a unique line which determines two points  $P_\infty, Q_\infty \in \mathbf{S}^n$ , say with  $P, Q, Q_\infty, P_\infty$  arranged in cyclic order on the euclidean circle defining this line. The (hyperbolic) distance between  $P$  and  $Q$  is given by a cross-ratio of euclidean distances; more precisely, it is defined to be

$$d(P, Q) = \text{Log}(P, Q, Q_\infty, P_\infty) = \log \left( \frac{|P - Q_\infty|}{|P - P_\infty|} : \frac{|Q - Q_\infty|}{|Q - P_\infty|} \right).$$

The *proper Möbius group*  $GM(n)_0$  is the group of orientation preserving isometries of  $\mathbf{R}^{n+1}$  for this distance. Any  $g \in GM(n)_0$  extends to a homeomorphism of the closed ball  $H^{n+1} \cup \mathbf{S}^n$ . One may check that  $GM(1)_0$  is isomorphic to  $PGL(2, \mathbf{R})$  and  $GM(2)_0$  to  $PGL(2, \mathbf{C})$ .

There is an equivalent description with  $H^{n+1}$  the half space  $\mathbf{R}^n \times \mathbf{R}_+^*$ . The set of “points at infinity” is then  $\mathbf{R}^n \cup \{\infty\}$  rather than  $\mathbf{S}^n$ .

For all this, see e.g. [A] or [Si].

An isometry  $g \in GM(n)_0$  is said to be

*elliptic* if there is some point in  $H^{n+1}$  fixed by  $g$ ,

*parabolic* if there is in  $\mathbf{S}^n$  exactly one point fixed by  $g$ ,

*hyperbolic* if there is a line in  $H^{n+1}$  invariant by  $g$  on which  $g$  has no fixed point.

(Following Thurston [Th], we call “hyperbolic” elements which are “loxodromic” in classical litterature, such as in [Gr].)

**PROPOSITION.** *Elliptic, parabolic and hyperbolic elements define a partition of the proper Möbius group in three disjoint classes.*

*Proof.* Let us first check that the three classes do not overlap in  $GM(n)_0$ . If  $g$  is hyperbolic, it has two fixed points in  $\mathbf{S}^n$  and thus cannot be parabolic; if  $g$  was