

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 29 (1983)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** REPRESENTATIONS OF THE SYMMETRIC GROUP, THE SPECIALIZATION ORDER, SYSTEMS AND GRASSMANN MANIFOLDS  
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**Kapitel:** 10. Deformations of representation homomorphisms and subrepresentations  
**DOI:** <https://doi.org/10.5169/seals-52973>

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So we have the following

- (1)  $t < i + p(i)$  (by hypothesis)
- (2)  $n - i \geq n + m - t - l$  or equivalently  $i \leq t + l - m$
- (3)  $\kappa_m + \dots + \kappa_{m-l+1} + l \leq n + m - t$
- (4)  $\kappa_1 + \dots + \kappa_{p(i)} < i \leq \kappa_1 + \dots + \kappa_{p(i)+1}$ .

Using (2) and (3) we have that

$$\kappa_m + \dots + \kappa_{m-l+1} \leq n - i = \kappa_1 + \dots + \kappa_m - i$$

so we have  $i \leq \kappa_1 + \dots + \kappa_{m-l}$  which implies  $m - l \geq p(i) + 1$  thus

$$p(i) + i \leq m - l - 1 + i \leq (m - l - 1) + (t + l - m) = t - 1$$

which contradicts (1). This proves the theorem.

**9.7. Vectorbundles and Schubert cells.** Because every positive vectorbundle over  $\mathbf{P}^1(\mathbf{C})$  arises as the bundle  $E(\Sigma)$  of some system  $\Sigma$  one has the obvious analogues of theorems 9.5 and 9.6 for positive bundles over  $\mathbf{P}^1(\mathbf{C})$ . Here the morphism  $\psi_\Sigma$  must, of course, be replaced by the classifying morphism (cf. section 3.2 above) of a positive vector bundle  $E$ , and  $n + m$  and  $m$  are determined respectively as  $\dim \Gamma(E, \mathbf{P}^1(\mathbf{C}))$  and  $\dim E$ .

## 10. DEFORMATIONS OF REPRESENTATION HOMOMORPHISMS AND SUBREPRESENTATIONS

**10.1 On proving Inclusion Results for Representations.** Suppose we have given a continuous family of homomorphisms of finite dimensional representations over  $\mathbf{C}$  of a finite group  $G$

$$(10.2) \quad \pi_t : M \rightarrow V$$

Suppose that  $\text{Im } \pi_t \simeq \rho$  for  $t \neq 0$  (and small) and that  $\text{Im } \pi_0 \simeq \rho_0$ . Then the representation  $\rho_0$  is a direct summand of the representation  $\rho$ . This is seen as follows. Because the category of finite dimensional complex representations of  $G$  is semisimple there is a homomorphism of representations  $\phi_0 : \text{Im } \pi_0 \rightarrow M$  such that  $\pi_0 \circ \phi_0 = \text{id}$ . Then  $\pi_t \circ \phi_0 : \text{Im } \phi_0 \rightarrow \text{Im } \pi_t$  is still injective for small  $t$  (by the continuity of  $\pi_t$ ) which gives us  $\rho_0$  as a subrepresentation and hence a direct summand of  $\rho$ .

It is almost equally easy to construct a surjective homomorphism  $\text{Im } \pi_t \rightarrow \text{Im } \pi_0$ .

10.3. *The Inverse Result.* Inversely if  $\rho_0$  is a subrepresentation of  $\rho$  then there is a family of representations (10.3) such that  $\text{Im } \pi_t \simeq \rho$  for  $t \neq 0$  and  $\text{Im } \pi_0 \simeq \rho_0$ , and if  $\rho$  is generated (as a  $\mathbf{C}[G]$ -module) by one element one can take for  $M$  in (10.2) the regular representation. Indeed if  $\rho_0$  is a subrepresentation of  $\rho$  then  $\rho = \rho_0 \oplus \rho_1$ . Let  $\pi: M \rightarrow \rho = \rho_0 \oplus \rho_1$  be a surjective map of representations. Let  $\pi_0, \pi_1$  be the two components of  $\pi$ . Let  $s = (s_0, s_1)$  be a section of  $\pi$ . Then  $\pi_0 s_0 = id, \pi_1 s_1 = id, \pi_0 s_1 = 0, \pi_1 s_0 = 0$  and it follows that  $\pi(t)$  consisting of the components  $\pi_0$  and  $t\pi_1$  is still surjective. Hence  $\text{Im } \pi(t) = \rho$  and  $\text{Im } \pi(0) = \rho_0$ .

## 11. A FAMILY OF REPRESENTATIONS OF $S_{n+m}$ PARAMETRIZED BY $\mathbf{G}_n(\mathbf{C}^{n+m})$

11.1. *Construction of the Family.* Let  $M$  be the regular representation of  $S_{n+m}$ . That is  $M$  has a basis  $e_\sigma, \sigma \in S_{n+m}$  and  $S_{n+m}$  acts on  $M$  by the formula  $\tau(e_\sigma) = e_{\tau\sigma}$ , for all  $\tau \in S_{n+m}$ . Now consider the universal bundle  $\xi_m$  over  $\mathbf{G}(\mathbf{C}^{n+m})$  and the  $n+m$  holomorphic section  $\varepsilon_1, \dots, \varepsilon_{n+m}$  defined by

$$\varepsilon_i(x) = e_i \bmod x \in \mathbf{C}^{n+m}/x,$$

where  $e_i$  is the  $i$ -th standard basis vector. Take the  $(m+n)$ -fold tensor product of  $\xi_m$  and define a family of homomorphisms parametrized by  $\mathbf{G}_n(\mathbf{C}^{n+m})$  by

$$(11.2) \quad \pi_x: M \rightarrow \xi_m(x)^{\otimes(n+m)}, e_{\sigma^{-1}} \mapsto \varepsilon_{\sigma(1)}(x) \otimes \dots \otimes \varepsilon_{\sigma(n)}(x)$$

More precisely (11.2) defines a homomorphism of vectorbundles

$$(11.3) \quad \mathbf{G}_n(\mathbf{C}^{n+m}) \times M \rightarrow \xi_m^{\otimes(n+m)}$$

The group  $S_{n+m}$  acts on  $\xi_m(x)^{\otimes(n+m)}$  by permuting the factors and it is a routine exercise to see that  $\pi_x$  is equivariant with respect to this action, i.e. that  $\pi_x(\tau v) = \tau \pi_x(v)$  for all  $v \in M, \tau \in S_{n+m}$ . (Here the product  $\tau\sigma \in S_{n+m}$  is interpreted as first the automorphism  $\sigma$  of  $1, \dots, n+m$  and then the automorphism  $\tau$ .)

Thus  $\text{Im } \pi_x = \pi(x)$  is a representation of  $S_{n+m}$  for all  $x$  giving us a family of representations parametrized by  $\mathbf{G}_n(\mathbf{C}^{n+m})$ . Fixing a point  $x_0 \in \mathbf{G}_n(\mathbf{C}^{n+m})$  and choosing  $m$  independent sections of  $\xi_m$  in a neighbourhood  $U$  of  $x_0$ , this gives us families of homomorphisms of representations