Zeitschrift: L'Enseignement Mathématique

Herausgeber: Commission Internationale de l'Enseignement Mathématique

Band: 29 (1983)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: REPRESENTATIONS OF THE SYMMETRIC GROUP, THE SPECIALIZATION ORDER, SYSTEMS AND GRASSMANN MANIFOLDS

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Kapitel: 9. Vectorbundles, systems and Schubert cells

DOI: https://doi.org/10.5169/seals-52973

i.e. mod $\psi_\Sigma(s)$ and for $s = 0$, $e_2 \equiv \dots \equiv e_{\kappa_1} \equiv e_{n+1} \equiv 0$ but $e_1 \neq 0$ and for $s = \infty$, $e_1 \equiv \dots \equiv e_{\kappa_1} \equiv 0$ and $e_{n+1} \neq 0$. It follows that the vectors

$$\varepsilon_1(\psi_\Sigma(s)), \dots, \varepsilon_{\kappa_1}(\psi_\Sigma(s)), \varepsilon_{n+1}(\psi_\Sigma(s))$$

span a one-dimensional subspace of $\xi_m(\psi_\Sigma(s))$ for all s so that $E(\Sigma) \simeq \psi_\Sigma^! \xi_m$ contains a line bundle L_1 which admits at least $\kappa_1 + 1$ linearly independent holomorphic sections viz. the $\varepsilon_1, \dots, \varepsilon_{\kappa_1}, \varepsilon_{n+1}$. Similar relations hold for

$$\varepsilon_{\kappa_1 + \dots + \kappa_{i-1} + 1}, \dots, \varepsilon_{\kappa_1 + \dots + \kappa_i}, \varepsilon_{n+1}$$

for all $i = 1, \dots, m$ giving us subbundles L_i, $i = 1, \dots, m$ which admit at least $\kappa_i + 1$ linearly independent holomorphic sections. This exhausts the ε_i and because the $\varepsilon_1(x), \dots, \varepsilon_{n+m}(x)$ span $\xi_m(x)$ for all $x \in \mathbf{G}_n(\mathbf{C}^{n+m})$ it follows that $E(\Sigma) = \oplus L_i$. As the pullback of the bundle ξ_m, $E(\Sigma)$ itself is a subbundle of an $(n+m)$-dimensional trivial bundle. Because $\mathbf{P}^1(\mathbf{C})$ is projective it follows (as before) that $E(\Sigma)$ has at most $n + m$ linearly independent holomorphic sections. But L_i has at least $\kappa_i + 1$ linearly independent sections, hence $\oplus L_i$ has at least $\Sigma(\kappa_i + 1) = n + m$ linearly independent sections which proves that L_i has precisely $\kappa_i + 1$ linearly independent sections and hence identifies L_i as the bundle $L(\kappa_i)$ described above in (8.5). We have reproved the theorem of Hermann and Martin [14].

8.12. *Theorem.* Keeping the notations introduced above in (8.10) and (8.5) we have $E(\Sigma) \simeq \bigoplus_{i=1}^{m} L(\kappa_i)$.

Still another proof of this theorem, using the Riemann-Roch theorem is found in Byrnes [33].

8.13. *The Correspondence B.* (cf. the diagram in section 5 above). The mapping $\Sigma \mapsto E(\Sigma)$ is obviously continuous. Thus the result $\overline{U(\kappa)} \supset U(\lambda) \leftrightarrow \kappa > \lambda$ can be deduced from Shatz's theorem (cf. 2.9). Inversely Shatz's theorem for positive bundles over $\mathbf{P}^1(\mathbf{C})$ can be deduced from the result on feedback orbits because every positive bundle arises as an $E(\Sigma)$. By tensoring with a suitable $L(r)$, r high enough, the result is then extended to arbitrary bundles over $\mathbf{P}^1(\mathbf{C})$.

9. Vectorbundles, systems and Schubert cells

9.1. *Partitions and Schubert-cells.* Let κ be a partition of n. To κ we associate the following increasing sequence of n numbers $\tau(\kappa)$.

(9.2)
$$\tau(\kappa) = (\underbrace{2, 3, \ldots, \kappa_1 + 1}_{\kappa_1}, \underbrace{\kappa_1 + 3, \ldots, \kappa_1 + \kappa_2 + 2}_{\kappa_2}, \ldots,$$
$$\underbrace{\kappa_1 + \ldots + \kappa_{m-1} + m + 1, \ldots, \kappa_1 + \ldots + \kappa_m + m)}_{\kappa_m}$$

Let $\tau_j(\kappa)$, $j = 1, \ldots, n$, be the j-th element of this sequence. It is an easy exercise to check that

(9.3)
$$\kappa > \lambda \leftrightarrow \tau_i(\kappa) \geqslant \tau_i(\lambda) \qquad \text{for all} \qquad i = 1, \ldots, n\,.$$

Thus the specialization order is a suborder of the inclusion ordering between closed Schubert cells, because

$$SC(\tau) \supset SC(\tau') \leftrightarrow \tau_i \geqslant \tau_i',\ i = 1, \ldots, n\,.$$

And in turn, as we saw above in section 4, the Schubert-cell order is a quotient of the Bruhat order on the Weyl group S_{n+m}.

9.4. *Systems and Schubert Cells.* Let $(A, B) \in L^{cr}_{m,n}$ be a system and as in section 8.8 consider the associated holomorphic morphism $\psi_\Sigma : \mathbf{P}^1(\mathbf{C}) \to \mathbf{G}_n(\mathbf{C}^{n+m})$. Suppose that (A, B) are in Brunovsky canonical form. Then simple inspection of the matrix $(sI - A; B)$ (cf. the example below proposition 8.11) shows that $Im\, \psi_\Sigma \subset SC(\tau(\kappa))$, where $\kappa = \kappa(A, B)$. Now let (A, B) be any system in $L^{cr}_{m,n}$. Then it is feedback equivalent to one in Brunovsky canonical form so that $(sI - A; B) = P(sI - A_0; B_0)Q$ for certain constant invertible matrices P, Q where (A_0, B_0) is a canonical pair. Premultiplication with P does not change ψ_Σ and postmultiplication with Q induces an automorphism of $\mathbf{G}_n(\mathbf{C}^{n+m})$ taking Schubert-cell $SC(\tau(\kappa))$ into another Schubert-cell of the same dimension type. Thus we have shown:

9.5. *Theorem.* Let $\Sigma \in L^{cr}_{m,n}$, $\kappa = \kappa(\Sigma)$ and let $\psi_\Sigma : \mathbf{P}^1(\mathbf{C}) \to \mathbf{G}_n(\mathbf{C}^{n+m})$ be the Hermann-Martin morphism of Σ. Then there is a Schubert-cell $SC(\underline{A})$, $\underline{A} = (A_1, \ldots, A_n)$ such that $Im\, \psi_\Sigma \subset SC(\underline{A})$ and $\dim A_i = \tau_i(\kappa)$, where $\tau_i(\kappa)$ is defined by (9.2).

We will now show that the Schubert-cell $SC(\underline{A})$ obtained in 9.5 is the smallest possible in the sense of the associated sequence of dimension numbers. We first prove a technical lemma.

9.6. *Lemma.* Let $\underline{X}(s)$ be the matrix, defined by a partition

$$\kappa_1 \geqslant \kappa_2 \geqslant \dots \geqslant \kappa_m, \kappa_1 + \dots + \kappa_m = n,$$

consisting of blocks $\underline{X}_i(s)$ where

$$\underline{X}_i(s) = \begin{bmatrix} s & -1 & & & 0 \\ & s & -1 & & 0 \\ & & \ddots & & \\ & & & -1 & \\ 0 & 0 & & s & 1 \end{bmatrix} \quad \kappa_i \times (\kappa_i + 1)$$

and

$$\underline{X}(s) = \begin{bmatrix} \underline{X}_1(s) & \ddots & 0 \\ 0 & & \underline{X}_1(s) \end{bmatrix} \quad n \times (n+m)$$

Let B be an $(m+n) \times \tau$ matrix of rank τ. Then $X(s)B$ has rank greater than or equal to $\tau - t$ for almost all s where t is the largest number such that

$$\kappa_m + \kappa_{m-1} + \dots + \kappa_{m-t+1} + t \leqslant \tau .$$

Proof. We first consider the case that there is only one κ, i.e., $m = 1$. We can assume that B is in column echelon form by postmultiplying by a nonsingular matrix if necessary. So B has the following form:

$$\begin{bmatrix} 0 & \dots & & 0 \\ I_{\lambda_1} & 0 & \dots & 0 \\ x & 0 & \dots & 0 \\ 0 & I_{\lambda_2} & & 0 \\ & & & \\ x & & x\,0 \dots & 0 \\ & \vdots & & \vdots \\ 0 & & \dots 0 & I_{\lambda_u} \\ x & & & x \end{bmatrix} \begin{matrix} r_1 \\ \lambda_1 \\ r_2 \\ \lambda_2 \\ \\ \\ \\ \lambda_u \\ r_{u+1} \end{matrix}$$

The x's stand for possibly nonzero blocks. Write

$$X(s) = s \begin{bmatrix} 1 & & & 0 \\ & \ddots & & \\ 0 & & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & & 0 \\ & \ddots & & \\ & & -1 & \\ 0 & & 0 & 1 \end{bmatrix} = sA_1 + A_2$$

and write $B = \begin{bmatrix} b_1 \\ \vdots \\ b_{n+1} \end{bmatrix}$ where b_i is the i-th row.

Now $X(s)B = \begin{bmatrix} sb_1 - b_2 \\ \vdots \\ sb_{n-1} - b_n \\ sb_n + b_{n+1} \end{bmatrix}$

We need to prove that $X(s)B$ has the required rank. Assume that B has rank τ and $\tau \leqslant n$. Let x be a τ vector and assume that

$$X(s)Bx = 0$$

We will show that either $x = 0$ or the equation only holds for finitely many values of s. We first note that

$$\begin{aligned} b_2x &= sb_1x \\ &\vdots \\ b_nx &= s^{n-1}b_1x \\ b_{n+1}x &= -s^n b_1x \end{aligned}$$

Thus if $b_1x = 0$ then $b_ix = 0$ for all x. But since B has full rank this implies that $x = 0$. Thus we may assume that $b_1x = 1$ and thus that $r_1 = 0$. So we have that $x_1 = 1, x_2 = s, ..., x_{\lambda_1} = s^{\lambda_1 - 1}$. If $r_2 = 0$, B is of the form $\begin{pmatrix} I_\tau \\ x \end{pmatrix}$ and the result is obvious, so we can assume $r_2 \neq 0$. Then we have

$$sb_{\lambda_1}x = b_{\lambda_1+1}x$$

so that

$$s^{\lambda_1} = b_{\lambda_1+1,1} + b_{\lambda_1+1,2}s + \dots + b_{\lambda_1+1,\lambda_1}s^{\lambda_1-1}$$

and this question is satisfied for only finitely many s. Therefore we have shown that if there is a nonzero solution of $X(s)Bx = 0$ then $b_1x \neq 0$ and the solution can exist only for finitely many values of s. Thus in this case the rank of $X(s)B$ is equal to τ for almost all s. If B is invertible (rank of B equal to $n + 1$) then the rank of $X(s)B$ is equal to $n =$ rank $X(s) =$ (rank B) $- 1$.

Now let m be greater than or equal to two. Again put B into column echelon form and partition B in such a way that the pieces $B_1, ..., B_m$ are still in column echelon form.

$$\begin{matrix} B_1 & 0 & \dots & 0 \\ x & B_2 & \dots & 0 \\ & \vdots & & \\ x & x & \dots & B_m \end{matrix} \qquad \begin{matrix} \kappa_1 + 1 \\ \kappa_2 + 1 \\ \\ \kappa_m + 1 \end{matrix}$$

The product $X(s)B$ has the form

$$\begin{matrix} X_1(s)B_1 & 0 & & \dots & 0 \\ ? & X_2(s)B_2 & 0 & \dots & 0 \\ & & \ddots & & \\ ? & & & & X_m(s)B_m \end{matrix}$$

It follows that the rank of $X(s)B$ is equal to the sum of the ranks of the $X_i(s)B_i$. From before we have that rank $X_i(s)B_i$ = rank B_i for all but finitely many s unless B_i is invertible in which case $X_i(s)B_i$ = rank $B_i - 1$. This proves the proposition. We can now prove the theorem that relates the ordering on the Schubert cells to the ordering on the orbits of the feedback group.

9.7. *Theorem.* Let (F, G) be a controllable pair and let ψ be the associated morphism from $\mathbf{P}^1(\mathbf{C})$ into $\mathbf{G}_n(\mathbf{C}^{n+m})$. Let $A_1 \dots A_n$ be a sequence of subspaces of $\mathbf{C}^{n+m}$ such that $\psi(\mathbf{P}^1(\mathbf{C}))$ is contained in the Schubert cell $SC(A_1, \dots, A_n)$. Let $\kappa_1, \dots, \kappa_m$ be the Kronecker indices of (F, G) and for each i let $p(i) = j$ iff

$$\kappa_1 + \dots + \kappa_j < i \leqslant \kappa_1 + \dots + \kappa_{j+1} .$$

Then dim $A_i \geqslant i + p(i) = \tau_i(\kappa)$.

Proof. It is a simple matter to check that $\tau_i(\kappa)$ (cf. (9.2) above) is equal to $i + p(i)$. We can assume that (F, G) is in Brunovsky canonical form. Suppose that dim $A_i = t < i + p(i)$. Then

$$A_i = \{x \in \mathbf{C}^{n+m} : \langle b_j, x \rangle = 0, j = 1, \dots, n + m - t\}$$

for certain linearly independent b_j. Let B be matrix whose columns are the b_i's. Let $P(s)$ be the space spanned by the rows of $X(s)$. Since $\psi(\mathbf{P}^1(\mathbf{C}))$ is contained in $SC(A_1, \dots, A_n)$ we must have that $\dim(A_i \cap P(s)) \geqslant i$. Thus the dimension of $P(s)B$ is less than or equal to $n - i$ which is the same as

$$\text{rank } X(s)B \leqslant n - i .$$

Now by the previous proposition rank $X(s)B \geqslant n + m - t - l$ where l is the largest number such that

$$\kappa_m + \kappa_{m-1} + \dots + \kappa_{m-l+1} + l \leqslant n + m - t .$$

So we have the following

(1) $t < i + p(i)$ (by hypothesis)

(2) $n - i \geqslant n + m - t - l$ or equivalently $i \leqslant t + l - m$

(3) $\kappa_m + \ldots + \kappa_{m-l+1} + l \leqslant n + m - t$

(4) $\kappa_1 + \ldots + \kappa_{p(i)} < i \leqslant \kappa_1 + \ldots + \kappa_{p(i)+1}$.

Using (2) and (3) we have that

$$\kappa_m + \ldots + \kappa_{m-l+1} \leqslant n - i = \kappa_1 + \ldots + \kappa_m - i$$

so we have $i \leqslant \kappa_1 + \ldots + \kappa_{m-l}$ which implies $m - l \geqslant p(i) + 1$ thus

$$p(i) + i \leqslant m - l - 1 + i \leqslant (m-l-1) + (t+l-m) = t - 1$$

which contradicts (1). This proves the theorem.

9.7. *Vectorbundles and Schubert cells.* Because every positive vectorbundle over $\mathbf{P}^1(\mathbf{C})$ arises as the bundle $E(\Sigma)$ of some system Σ one has the obvious analogues of theorems 9.5 and 9.6 for positive bundles over $\mathbf{P}^1(\mathbf{C})$. Here the morphism ψ_Σ must, of course, be replaced by the classifying morphism (cf. section 3.2 above) of a positive vector bundle E, and $n + m$ and m are determined respectively as dim $\Gamma(E, \mathbf{P}^1(\mathbf{C}))$ and dim E.

10. Deformations of representation homomorphisms and subrepresentations

10.1 *On proving Inclusion Results for Representations.* Suppose we have given a continuous family of homomorphisms of finite dimensional representations over $\mathbf{C}$ of a finite group G

(10.2) $$\pi_t : M \to V$$

Suppose that $Im\, \pi_t \simeq \rho$ for $t \neq 0$ (and small) and that $Im\, \pi_0 \simeq \rho_0$. Then the representation ρ_0 is a direct summand of the representation ρ. This is seen as follows. Because the category of finite dimensional complex representations of G is semisimple there is a homomorphism of representations $\phi_0 : Im\, \pi_0 \to M$ such that $\pi_0 \circ \phi_0 = id$. Then $\pi_t \circ \phi_0 : Im\, \phi_0 \to Im\, \pi_t$ is still injective for small t (by the continuity of π_t) which gives us ρ_0 as a subrepresentation and hence a direct summand of ρ.

It is almost equally easy to construct a surjective homomorphism $Im\, \pi_t \to Im\, \pi_0$.