

Zeitschrift:	L'Enseignement Mathématique
Herausgeber:	Commission Internationale de l'Enseignement Mathématique
Band:	29 (1983)
Heft:	1-2: L'ENSEIGNEMENT MATHÉMATIQUE
 Artikel:	REPRESENTATIONS OF THE SYMMETRIC GROUP, THE SPECIALIZATION ORDER, SYSTEMS AND GRASSMANN MANIFOLDS
Autor:	Hazewinkel, Michiel / Martin, Clyde F.
Kapitel:	9. Vectorbundles, systems and Schubert cells
DOI:	https://doi.org/10.5169/seals-52973

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 11.02.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

i.e. mod $\psi_\Sigma(s)$ and for $s = 0$, $e_2 \equiv \dots \equiv e_{\kappa_1} \equiv e_{n+1} \equiv 0$ but $e_1 \neq 0$ and for $s = \infty$, $e_1 \equiv \dots \equiv e_{\kappa_1} \equiv 0$ and $e_{n+1} \neq 0$. It follows that the vectors

$$\varepsilon_1(\psi_\Sigma(s)), \dots, \varepsilon_{\kappa_1}(\psi_\Sigma(s)), \varepsilon_{n+1}(\psi_\Sigma(s))$$

span a one-dimensional subspace of $\xi_m(\psi_\Sigma(s))$ for all s so that $E(\Sigma) \simeq \psi_\Sigma^! \xi_m$ contains a line bundle L_1 which admits at least $\kappa_1 + 1$ linearly independent holomorphic sections viz. the $\varepsilon_1, \dots, \varepsilon_{\kappa_1}, \varepsilon_{n+1}$. Similar relations hold for

$$\varepsilon_{\kappa_1 + \dots + \kappa_{i-1} + 1}, \dots, \varepsilon_{\kappa_1 + \dots + \kappa_i}, \varepsilon_{n+1}$$

for all $i = 1, \dots, m$ giving us subbundles L_i , $i = 1, \dots, m$ which admit at least $\kappa_i + 1$ linearly independent holomorphic sections. This exhausts the ε_i and because the $\varepsilon_1(x), \dots, \varepsilon_{n+m}(x)$ span $\xi_m(x)$ for all $x \in \mathbf{G}_n(\mathbf{C}^{n+m})$ it follows that $E(\Sigma) = \bigoplus L_i$. As the pullback of the bundle ξ_m , $E(\Sigma)$ itself is a subbundle of an $(n+m)$ -dimensional trivial bundle. Because $\mathbf{P}^1(\mathbf{C})$ is projective it follows (as before) that $E(\Sigma)$ has at most $n + m$ linearly independent holomorphic sections. But L_i has at least $\kappa_i + 1$ linearly independent sections, hence $\bigoplus L_i$ has at least $\Sigma(\kappa_i + 1) = n + m$ linearly independent sections which proves that L_i has precisely $\kappa_i + 1$ linearly independent sections and hence identifies L_i as the bundle $L(\kappa_i)$ described above in (8.5). We have reproved the theorem of Hermann and Martin [14].

8.12. *Theorem.* Keeping the notations introduced above in (8.10) and (8.5) we have $E(\Sigma) \simeq \bigoplus_{i=1}^m L(\kappa_i)$.

Still another proof of this theorem, using the Riemann-Roch theorem is found in Byrnes [33].

8.13. *The Correspondence B.* (cf. the diagram in section 5 above). The mapping $\Sigma \mapsto E(\Sigma)$ is obviously continuous. Thus the result $\overline{U(\kappa)} \supset U(\lambda) \leftrightarrow \kappa > \lambda$ can be deduced from Shatz's theorem (cf. 2.9). Inversely Shatz's theorem for positive bundles over $\mathbf{P}^1(\mathbf{C})$ can be deduced from the result on feedback orbits because every positive bundle arises as an $E(\Sigma)$. By tensoring with a suitable $L(r)$, r high enough, the result is then extended to arbitrary bundles over $\mathbf{P}^1(\mathbf{C})$.

9. VECTORBUNDLES, SYSTEMS AND SCHUBERT CELLS

9.1. *Partitions and Schubert-cells.* Let κ be a partition of n . To κ we associate the following increasing sequence of n numbers $\tau(\kappa)$.

$$(9.2) \quad \tau(\kappa) = (\underbrace{2, 3, \dots, \kappa_1 + 1}_{\kappa_1}, \underbrace{\kappa_1 + 3, \dots, \kappa_1 + \kappa_2 + 2, \dots}_{\kappa_2}, \dots, \underbrace{\kappa_1 + \dots + \kappa_{m-1} + m + 1, \dots, \kappa_1 + \dots + \kappa_m + m}_{\kappa_m})$$

Let $\tau_j(\kappa)$, $j = 1, \dots, n$, be the j -th element of this sequence. It is an easy exercise to check that

$$(9.3) \quad \kappa > \lambda \leftrightarrow \tau_i(\kappa) \geq \tau_i(\lambda) \quad \text{for all } i = 1, \dots, n.$$

Thus the specialization order is a suborder of the inclusion ordering between closed Schubert cells, because

$$SC(\tau) \supset SC(\tau') \leftrightarrow \tau_i \geq \tau'_i, i = 1, \dots, n.$$

And in turn, as we saw above in section 4, the Schubert-cell order is a quotient of the Bruhat order on the Weyl group S_{n+m} .

9.4. Systems and Schubert Cells. Let $(A, B) \in L_{m,n}^{cr}$ be a system and as in section 8.8 consider the associated holomorphic morphism $\psi_\Sigma : \mathbf{P}^1(\mathbf{C}) \rightarrow \mathbf{G}_n(\mathbf{C}^{n+m})$. Suppose that (A, B) are in Brunovsky canonical form. Then simple inspection of the matrix $(sI - A; B)$ (cf. the example below proposition 8.11) shows that $Im \psi_\Sigma \subset SC(\tau(\kappa))$, where $\kappa = \kappa(A, B)$. Now let (A, B) be any system in $L_{m,n}^{cr}$. Then it is feedback equivalent to one in Brunovsky canonical form so that $(sI - A; B) = P(sI - A_0; B_0)Q$ for certain constant invertible matrices P, Q where (A_0, B_0) is a canonical pair. Premultiplication with P does not change ψ_Σ and postmultiplication with Q induces an automorphism of $\mathbf{G}_n(\mathbf{C}^{n+m})$ taking Schubert-cell $SC(\tau(\kappa))$ into another Schubert-cell of the same dimension type. Thus we have shown:

9.5. Theorem. Let $\Sigma \in L_{m,n}^{cr}$, $\kappa = \kappa(\Sigma)$ and let $\psi_\Sigma : \mathbf{P}^1(\mathbf{C}) \rightarrow \mathbf{G}_n(\mathbf{C}^{n+m})$ be the Hermann-Martin morphism of Σ . Then there is a Schubert-cell $SC(\underline{A})$, $\underline{A} = (A_1, \dots, A_n)$ such that $Im \psi_\Sigma \subset SC(\underline{A})$ and $\dim A_i = \tau_i(\kappa)$, where $\tau_i(\kappa)$ is defined by (9.2).

We will now show that the Schubert-cell $SC(\underline{A})$ obtained in 9.5 is the smallest possible in the sense of the associated sequence of dimension numbers. We first prove a technical lemma.

9.6. *Lemma.* Let $\underline{X}(s)$ be the matrix, defined by a partition

$$\kappa_1 \geq \kappa_2 \geq \dots \geq \kappa_m, \kappa_1 + \dots + \kappa_m = n,$$

consisting of blocks $\underline{X}_i(s)$ where

$$\underline{X}_i(s) = \begin{bmatrix} s & -1 & & 0 \\ & s & -1 & 0 \\ & & \ddots & \\ & & & -1 \\ 0 & 0 & & s & 1 \end{bmatrix} \quad \kappa_i \times (\kappa_i + 1)$$

and

$$\underline{X}(s) = \begin{bmatrix} \underline{X}_1(s) & & 0 \\ 0 & \ddots & \underline{X}_1(s) \end{bmatrix} \quad n \times (n+m)$$

Let B be an $(m+n) \times \tau$ matrix of rank τ . Then $\underline{X}(s)B$ has rank greater than or equal to $\tau - t$ for almost all s where t is the largest number such that

$$\kappa_m + \kappa_{m-1} + \dots + \kappa_{m-t+1} + t \leq \tau.$$

Proof. We first consider the case that there is only one κ , i.e., $m = 1$. We can assume that B is in column echelon form by postmultiplying by a nonsingular matrix if necessary. So B has the following form:

$$\begin{bmatrix} 0 & \dots & 0 & r_1 \\ I_{\lambda_1} & 0 & \dots & 0 & \lambda_1 \\ x & 0 & \dots & 0 & r_2 \\ 0 & I_{\lambda_2} & & 0 & \lambda_2 \\ & & & & \\ x & x & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & I_{\lambda_u} & \lambda_u \\ x & & & x & r_{u+1} \end{bmatrix}$$

The x 's stand for possibly nonzero blocks. Write

$$X(s) = s \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & & 0 \\ & \ddots & & -1 \\ 0 & & 0 & 1 \end{bmatrix} = sA_1 + A_2$$

and write $B = \begin{bmatrix} b_1 \\ \vdots \\ b_{n+1} \end{bmatrix}$ where b_i is the i -th row.

Now $X(s)B = \begin{bmatrix} sb_1 - b_2 \\ \vdots \\ sb_{n-1} - b_n \\ sb_n + b_{n+1} \end{bmatrix}$

We need to prove that $X(s)B$ has the required rank. Assume that B has rank τ and $\tau \leq n$. Let x be a τ vector and assume that

$$X(s)Bx = 0$$

We will show that either $x = 0$ or the equation only holds for finitely many values of s . We first note that

$$\begin{aligned} b_2x &= sb_1x \\ &\vdots \\ b_nx &= s^{n-1}b_1x \\ b_{n+1}x &= -s^n b_1x \end{aligned}$$

Thus if $b_1x = 0$ then $b_i x = 0$ for all x . But since B has full rank this implies that $x = 0$. Thus we may assume that $b_1x = 1$ and thus that $r_1 = 0$. So we have that $x_1 = 1, x_2 = s, \dots, x_{\lambda_1} = s^{\lambda_1-1}$. If $r_2 = 0$, B is of the form $\begin{pmatrix} I_\tau \\ x \end{pmatrix}$ and the result is obvious, so we can assume $r_2 \neq 0$. Then we have

$$sb_{\lambda_1}x = b_{\lambda_1+1}x$$

so that

$$s^{\lambda_1} = b_{\lambda_1+1, 1} + b_{\lambda_1+1, 2}s + \dots + b_{\lambda_1+1, \lambda_1}s^{\lambda_1-1}$$

and this question is satisfied for only finitely many s . Therefore we have shown that if there is a nonzero solution of $X(s)Bx = 0$ then $b_1x \neq 0$ and the solution can exist only for finitely many values of s . Thus in this case the rank of $X(s)B$ is equal to τ for almost all s . If B is invertible (rank of B equal to $n + 1$) then the rank of $X(s)B$ is equal to $n = \text{rank } X(s) = (\text{rank } B) - 1$.

Now let m be greater than or equal to two. Again put B into column echelon form and partition B in such a way that the pieces B_1, \dots, B_m are still in column echelon form.

$$\begin{array}{ccccc}
 B_1 & 0 & \dots & 0 & \kappa_1 + 1 \\
 x & B_2 & \dots & 0 & \kappa_2 + 1 \\
 \vdots & & & & \\
 x & x & \dots & B_m & \kappa_m + 1
 \end{array}$$

The product $X(s)B$ has the form

$$\begin{array}{ccccc}
 X_1(s)B_1 & 0 & \dots & 0 & \\
 ? & X_2(s)B_2 & 0 & \dots & 0 \\
 & & \ddots & & \\
 ? & & & & X_m(s)B_m
 \end{array}$$

It follows that the rank of $X(s)B$ is equal to the sum of the ranks of the $X_i(s)B_i$. From before we have that $\text{rank } X_i(s)B_i = \text{rank } B_i$ for all but finitely many s unless B_i is invertible in which case $X_i(s)B_i = \text{rank } B_i - 1$. This proves the proposition. We can now prove the theorem that relates the ordering on the Schubert cells to the ordering on the orbits of the feedback group.

9.7. *Theorem.* Let (F, G) be a controllable pair and let ψ be the associated morphism from $\mathbf{P}^1(\mathbf{C})$ into $\mathbf{G}_n(\mathbf{C}^{n+m})$. Let $A_1 \dots A_n$ be a sequence of subspaces of \mathbf{C}^{n+m} such that $\psi(\mathbf{P}^1(\mathbf{C}))$ is contained in the Schubert cell $SC(A_1, \dots, A_n)$. Let $\kappa_1, \dots, \kappa_m$ be the Kronecker indices of (F, G) and for each i let $p(i) = j$ iff

$$\kappa_1 + \dots + \kappa_j < i \leq \kappa_1 + \dots + \kappa_{j+1}.$$

Then $\dim A_i \geq i + p(i) = \tau_i(\kappa)$.

Proof. It is a simple matter to check that $\tau_i(\kappa)$ (cf. (9.2) above) is equal to $i + p(i)$. We can assume that (F, G) is in Brunovsky canonical form. Suppose that $\dim A_i = t < i + p(i)$. Then

$$A_i = \{x \in \mathbf{C}^{n+m} : \langle b_j, x \rangle = 0, j = 1, \dots, n+m-t\}$$

for certain linearly independent b_j . Let B be matrix whose columns are the b_i 's. Let $P(s)$ be the space spanned by the rows of $X(s)$. Since $\psi(\mathbf{P}^1(\mathbf{C}))$ is contained in $SC(A_1, \dots, A_n)$ we must have that $\dim(A_i \cap P(s)) \geq i$. Thus the dimension of $P(s)B$ is less than or equal to $n - i$ which is the same as

$$\text{rank } X(s)B \leq n - i.$$

Now by the previous proposition $\text{rank } X(s)B \geq n + m - t - l$ where l is the largest number such that

$$\kappa_m + \kappa_{m-1} + \dots + \kappa_{m-l+1} + l \leq n + m - t.$$

So we have the following

- (1) $t < i + p(i)$ (by hypothesis)
- (2) $n - i \geq n + m - t - l$ or equivalently $i \leq t + l - m$
- (3) $\kappa_m + \dots + \kappa_{m-l+1} + l \leq n + m - t$
- (4) $\kappa_1 + \dots + \kappa_{p(i)} < i \leq \kappa_1 + \dots + \kappa_{p(i)+1}$.

Using (2) and (3) we have that

$$\kappa_m + \dots + \kappa_{m-l+1} \leq n - i = \kappa_1 + \dots + \kappa_m - i$$

so we have $i \leq \kappa_1 + \dots + \kappa_{m-l}$ which implies $m - l \geq p(i) + 1$ thus

$$p(i) + i \leq m - l - 1 + i \leq (m - l - 1) + (t + l - m) = t - 1$$

which contradicts (1). This proves the theorem.

9.7. *Vectorbundles and Schubert cells.* Because every positive vectorbundle over $\mathbf{P}^1(\mathbf{C})$ arises as the bundle $E(\Sigma)$ of some system Σ one has the obvious analogues of theorems 9.5 and 9.6 for positive bundles over $\mathbf{P}^1(\mathbf{C})$. Here the morphism ψ_Σ must, of course, be replaced by the classifying morphism (cf. section 3.2 above) of a positive vector bundle E , and $n + m$ and m are determined respectively as $\dim \Gamma(E, \mathbf{P}^1(\mathbf{C}))$ and $\dim E$.

10. DEFORMATIONS OF REPRESENTATION HOMOMORPHISMS AND SUBREPRESENTATIONS

10.1 *On proving Inclusion Results for Representations.* Suppose we have given a continuous family of homomorphisms of finite dimensional representations over \mathbf{C} of a finite group G

$$(10.2) \quad \pi_t : M \rightarrow V$$

Suppose that $\text{Im } \pi_t \simeq \rho$ for $t \neq 0$ (and small) and that $\text{Im } \pi_0 \simeq \rho_0$. Then the representation ρ_0 is a direct summand of the representation ρ . This is seen as follows. Because the category of finite dimensional complex representations of G is semisimple there is a homomorphism of representations $\phi_0 : \text{Im } \pi_0 \rightarrow M$ such that $\pi_0 \circ \phi_0 = \text{id}$. Then $\pi_t \circ \phi_0 : \text{Im } \phi_0 \rightarrow \text{Im } \pi_t$ is still injective for small t (by the continuity of π_t) which gives us ρ_0 as a subrepresentation and hence a direct summand of ρ .

It is almost equally easy to construct a surjective homomorphism $\text{Im } \pi_t \rightarrow \text{Im } \pi_0$.