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Artikel: REPRESENTATIONS OF THE SYMMETRIC GROUP, THE SPECIALIZATION ORDER, SYSTEMS AND GRASSMANN MANIFOLDS

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Then using the results above one shows that

$$\underline{t}\,\underline{s}(\overline{0(\kappa)}) = \overline{0(\kappa)},\ \underline{s}\,\underline{t}(\overline{U(\kappa)}) = \overline{U(\kappa)}$$

so that $\underline{t}$ and $\underline{s}$ set up a bijective correspondence between the closures of orbits in the two cases and hence induce a bijective order preserving correspondence between the sets of orbits themselves.

8. VECTORBUNDLES AND SYSTEMS

This section contains a modified version of the construction of Hermann-Martin [14] associating a vectorbundle $E(\Sigma)$ over the Riemann sphere $\mathbf{P}^1(\mathbf{C})$ to every $\Sigma = (A, B) \in L^{cr}_{m,n}$. This version makes it almost trivial to see that $E(\Sigma)$ splits as a direct sum of line bundles $L(\kappa_i)$, $i = 1, \ldots, m$ where $\kappa = (\kappa_1, \ldots, \kappa_m)$ is the set of Kronecker indices of Σ.

The first thing needed is some more information on the universal bundle ξ_m.

8.1. *On the Universal Bundle* $\xi_m \to \mathbf{G}_n(\mathbf{C}^{n+m})$. Let V be a complex $n + m$ dimensional vector space and $V^* = \mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})$ its dual vector space. Given $x \in \mathbf{G}_n(\mathbf{C}^{n+m})$ define $x^* = \{y \in V^* \mid < y, v > = 0 \text{ for all } x \in V\}$ where $< , >$ denotes the usual pairing $V^* \times V \to \mathbf{C}$. Then x^* is m-dimensional and $x \mapsto x^*$ defines a holomorphic isomorphism

$$d : \mathbf{G}_n(V) \to \mathbf{G}_m(V^*)\,. \tag{8.2}$$

Now $v \in V/x$ defines a unique homomorphism $v^T : x^* \to \mathbf{C}$ as follows:

$v^T(a) = < a, \tilde{v} >$ for all $a \in x^*$, where $\tilde{v} \in V$ is any representative of v. This is well defined because $< a, b > = 0$ for all $b \in x$ if $a \in x^*$. This defines an isomorphism between the pullback $(d^{-1})\ \xi_m$ and the dual of the subbundle η_m on $G_m(V^*)$ defined by

$$\eta_m = \{(x^*, w) \in \mathbf{G}_m(V^*) \times V^* \mid w \in x^*\}$$

It follows that ξ_m is a subbundle of an $n + m$ dimensional trivial bundle $\mathbf{G}_n(\mathbf{C}^{n+m}) \times \mathbf{C}^{n+m}$. Because $\mathbf{G}_n(\mathbf{C}^{n+m})$ is projective (compact) all holomorphic maps $\mathbf{G}_n(\mathbf{C}^{n+m}) \to \mathbf{C}$ are constant so that the space of holomorphic sections $\Gamma(\mathbf{G}_n(\mathbf{C}^{n+m}) \times \mathbf{C}^{n+m}, \mathbf{G}_n(\mathbf{C}^{n+m}))$ is of dimension $n + m$. As a subbundle of a trivial $(n+m)$-dimensional bundle ξ_m can therefore have at most $(n+m)$ linearly independent holomorphic sections. But we have already found $(n+m)$ linearly independent sections viz. the $\varepsilon_1, \ldots, \varepsilon_{n+m}$ defined by $\varepsilon_i(x) = e_i \bmod x$ where e_i is the i-th standard basis vector of $\mathbf{C}^{n+m}$. Therefore

$$\dim \Gamma(\xi_m, \mathbf{G}_n(\mathbf{C}^{n+m})) = n + m \tag{8.3}$$

Now let $A \in \mathbf{GL}_{n+m}(\mathbf{C})$. Then A induces a holomorphic automorphism A^* of $\mathbf{G}_m(\mathbf{C}^{n+m})$ defined by $x \mapsto Ax$. Then, of course, there is an induced isomorphism $A^{-1} : \mathbf{C}^{n+m}/Ax \to \mathbf{C}^{n+m}/x$ which for varying x induces an isomorphism

$$A^{!}_{*}\xi_m \simeq \xi_m,\ A \in \mathbf{GL}_{n+m}(\mathbf{C}) \tag{8.4}$$

8.5. *The Line Bundles* $L(i)$ *over* $\mathbf{P}^1(\mathbf{C})$. The Riemann sphere $\mathbf{P}^1(\mathbf{C}) = S^2$ can be obtained by gluing together two copies of $\mathbf{C}$ along the open subsets $\mathbf{C}\backslash\{0\}$ by the isomorphism

$$\mathbf{C}\backslash\{0\} \to \mathbf{C}\backslash\{0\},\ s \mapsto t = s^{-1}$$

A line bundle over $\mathbf{P}^1(\mathbf{C})$ is then obtained by giving a holomorphic isomorphism $\mathbf{C}\backslash\{0\} \times \mathbf{C} \to \mathbf{C}\backslash\{0\} \times \mathbf{C}$ linear in the second variable compatible with the above isomorphism. Obviously the only possibilities are $(s, v) \to (s^{-1}, s^i v)$ for $i \in \mathbf{Z}$. This gives us the following commutative diagram identifications

$$\begin{tikzcd}
\mathbf{C}\times\mathbf{C} \arrow[r, phantom, "\supset"] \arrow[d] & \mathbf{C}\backslash\{0\}\times\mathbf{C} \arrow[rr, "{(s,v)\to(s^{-1},s^iv)}"] \arrow[d] & & \mathbf{C}\backslash\{0\}\times\mathbf{C} \arrow[r, phantom, "\subset"] \arrow[d] & \mathbf{C}\times\mathbf{C} \arrow[d] \\
\mathbf{C} \arrow[r, phantom, "\supset"] \arrow[u, dashed, bend left, "s_1"] & \mathbf{C}\backslash\{0\} \arrow[rr, "s\to s^{-1}=t"'] & & \mathbf{C}\backslash\{0\} \arrow[r, phantom, "\subset"] & \mathbf{C} \arrow[u, dashed, bend right, "s_2"']
\end{tikzcd}$$

The corresponding holomorphic line bundle is denoted $L(-i)$. A section of $L(-i)$ consists of two holomorphic mappings s_1, s_2 of the form $s \to (s, f(s))$, $t \to (t, g(t))$ such that $s^i f(s) = g(s^{-1})$. It readily follows that $f(s)$ must be a polynomial of degree $\leqslant -i$. Thus

$$\dim \Gamma(L(i)) = 0 \qquad \text{if } i < 0 \tag{8.6}$$

$$\dim \Gamma(L(i)) = i + 1 \qquad \text{if } i \geqslant 0 \tag{8.7}$$

8.8. *The (modified) Hermann-Martin vectorbundle of a system.* Let $\Sigma = (A, B)$ be a pair of real or complex matrices of sizes $n \times n$ and $n \times m$. Then (A, B) is completely reachable (*cr*) iff the $n \times (n+m)$ matrix $(sI - A\,;\, B)$ is of rank n for all complex values of s. So if $\Sigma = (A, B)$ is *cr* one can define a holomorphic map ψ_Σ by

(8.9) $\quad \psi_\Sigma : \mathbf{P}^1(\mathbf{C}) \to \mathbf{G}_n(\mathbf{C}^{n+m}),\ s \mapsto \mathrm{Row}(sI-A\,;B),\ \infty \mapsto \mathrm{Row}(I\,;0)$

where $\mathrm{Row}(M)$ for an $n \times (m+n)$ matrix M denotes the subspace of $\mathbf{C}^{n+m}$ generated by the rows of M. The vectorbundle $E(\Sigma)$ over $\mathbf{P}^1(\mathbf{C})$ is now defined by

(8.10) $$E(\Sigma) = \psi_\Sigma^! \xi_m$$

8.11. *Proposition.* $E(\Sigma)$ depends only on the feedback orbit of Σ.

Indeed one easily checks that $\Sigma = (A, B)$, $\Sigma' = (A', B') \in L^{cr}_{m,n}$ are feedback equivalent (cf. 2.6 above) iff there are constant invertible matrices P, Q such that

$$P(sI-A\,;B)Q = (sI-A'\,;B')\,.$$

Now $\mathrm{Row}(PM) = \mathrm{Row}(M)$ and postmultiplication with Q changes ψ_Σ to $Q_* \circ \psi_\Sigma$ and

$$E(\Sigma') = \psi_\Sigma^!(\xi_m) = \psi_\Sigma^!(Q_*^!\,\xi_m) \simeq (\psi_\Sigma^!(\xi_m) = E(\Sigma))$$

by 8.4 above, proving the proposition.

Thus to determine $E(\Sigma)$ we can assume that $\Sigma = (A, B)$ is in Brunowsky canonical form which means that A, B takes the form

$$\left[\begin{array}{c|c|c} \begin{matrix}0&1&0\\&&\\&&1\\0&&0\end{matrix} & 0 & 0 \\ \hline 0 & \begin{matrix}0&1&0\\&&\\&&1\\0&&0\end{matrix} & 0 \\ \hline 0 & 0 & \begin{matrix}0&1&0\\&&\\&&1\\0&&0\end{matrix} \end{array}\right] \quad \left[\begin{array}{c} \begin{matrix}0&0&0\\&&\\0&&\\1&0&0\end{matrix} \\ \hline \begin{matrix}0&0&0\\&&\\&0&\\0&1&0\end{matrix} \\ \hline \begin{matrix}0&0&0\\&&\\&&0\\0&0&1\end{matrix} \end{array}\right] \quad \begin{array}{c} \kappa_1 \\ \\ \\ \\ \kappa_2 \\ \\ \\ \\ \kappa_3 \end{array}$$

in case $m = 3$, where $(\kappa_1, \kappa_2, \kappa_3) = \kappa(A, B)$ are the Kronecker indices of $\Sigma = (A, B)$. (The general case is evident from this example); every $(A, B) \in U(\kappa)$ is feedback equivalent to such a pair [30, 9]. The matrix $(sI-A\,;B)$ is now easily written down, and one observes that for all

$$s \neq 0, \infty, e_1 \equiv e_2 \equiv \dots \equiv e_{\kappa_1} \equiv e_{n+1} \bmod \mathrm{Row}(sI-A\,;B)\,,$$

i.e. mod $\psi_\Sigma(s)$ and for $s = 0$, $e_2 \equiv \dots \equiv e_{\kappa_1} \equiv e_{n+1} \equiv 0$ but $e_1 \neq 0$ and for $s = \infty$, $e_1 \equiv \dots \equiv e_{\kappa_1} \equiv 0$ and $e_{n+1} \neq 0$. It follows that the vectors

$$\varepsilon_1(\psi_\Sigma(s)), \dots, \varepsilon_{\kappa_1}(\psi_\Sigma(s)), \varepsilon_{n+1}(\psi_\Sigma(s))$$

span a one-dimensional subspace of $\xi_m(\psi_\Sigma(s))$ for all s so that $E(\Sigma) \simeq \psi_\Sigma^! \xi_m$ contains a line bundle L_1 which admits at least $\kappa_1 + 1$ linearly independent holomorphic sections viz. the $\varepsilon_1, \dots, \varepsilon_{\kappa_1}, \varepsilon_{n+1}$. Similar relations hold for

$$\varepsilon_{\kappa_1 + \dots + \kappa_{i-1} + 1}, \dots, \varepsilon_{\kappa_1 + \dots + \kappa_i}, \varepsilon_{n+1}$$

for all $i = 1, \dots, m$ giving us subbundles L_i, $i = 1, \dots, m$ which admit at least $\kappa_i + 1$ linearly independent holomorphic sections. This exhausts the ε_i and because the $\varepsilon_1(x), \dots, \varepsilon_{n+m}(x)$ span $\xi_m(x)$ for all $x \in \mathbf{G}_n(\mathbf{C}^{n+m})$ it follows that $E(\Sigma) = \oplus L_i$. As the pullback of the bundle ξ_m, $E(\Sigma)$ itself is a subbundle of an $(n+m)$-dimensional trivial bundle. Because $\mathbf{P}^1(\mathbf{C})$ is projective it follows (as before) that $E(\Sigma)$ has at most $n + m$ linearly independent holomorphic sections. But L_i has at least $\kappa_i + 1$ linearly independent sections, hence $\oplus L_i$ has at least $\Sigma(\kappa_i + 1) = n + m$ linearly independent sections which proves that L_i has precisely $\kappa_i + 1$ linearly independent sections and hence identifies L_i as the bundle $L(\kappa_i)$ described above in (8.5). We have reproved the theorem of Hermann and Martin [14].

8.12. *Theorem.* Keeping the notations introduced above in (8.10) and (8.5) we have $E(\Sigma) \simeq \bigoplus_{i=1}^{m} L(\kappa_i)$.

Still another proof of this theorem, using the Riemann-Roch theorem is found in Byrnes [33].

8.13. *The Correspondence B.* (cf. the diagram in section 5 above). The mapping $\Sigma \mapsto E(\Sigma)$ is obviously continuous. Thus the result $\overline{U(\kappa)} \supset U(\lambda) \leftrightarrow \kappa > \lambda$ can be deduced from Shatz's theorem (cf. 2.9). Inversely Shatz's theorem for positive bundles over $\mathbf{P}^1(\mathbf{C})$ can be deduced from the result on feedback orbits because every positive bundle arises as an $E(\Sigma)$. By tensoring with a suitable $L(r)$, r high enough, the result is then extended to arbitrary bundles over $\mathbf{P}^1(\mathbf{C})$.

9. Vectorbundles, systems and Schubert cells

9.1. *Partitions and Schubert-cells.* Let κ be a partition of n. To κ we associate the following increasing sequence of n numbers $\tau(\kappa)$.