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Artikel: REPRESENTATIONS OF THE SYMMETRIC GROUP, THE SPECIALIZATION ORDER, SYSTEMS AND GRASSMANN MANIFOLDS

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the point of $G_n(\mathbf{C}^{n+m})$ represented by $\text{Ker}(V \rightarrow E(x))$. This gives a holomorphic map $\Psi_E: M \rightarrow G_n(\mathbf{C}^{n+m})$ such that the pullback of ξ_m by means of Ψ_E is isomorphic to E , $\Psi_E^! \xi_m \simeq E$. It is universality properties such as this one which account for the importance of the bundles ξ_m and η_n in differential and algebraic topology [16], algebraic geometry and also system and control theory (cf. [22, 23] and the references therein for the last mentioned).

The bundle ξ_m has a number of obvious holomorphic sections, viz. the sections defined by $\varepsilon_i(x) = e_i \bmod x$ where e_i is the i -th standard basis vector of \mathbf{C}^{n+m} , $i = 1, \dots, n+m$. And, as a matter of fact, it is not difficult to show that $\Gamma(\xi_m, G_n(\mathbf{C}^{n+m}))$ is $(n+m)$ -dimensional and that the $\varepsilon_1, \dots, \varepsilon_{n+m}$ form a basis for this space of holomorphic sections; cf. subsection 8.1 below.

4. SCHUBERT CELLS

4.1. *Schubert Cells.* Consider again the Grassmann manifold $G_n(\mathbf{C}^{m+n})$. Let $\underline{A} = (A_1, \dots, A_n)$ be a sequence of n -subspaces of \mathbf{C}^{n+m} such that $0 \neq A_1 \subset A_2 \subset \dots \subset A_n$ with each containment strict. To each such sequence \underline{A} we associate the closed subset

$$(4.2) \quad SC(\underline{A}) = \{x \in G_n(\mathbf{C}^{m+n}) \mid \dim(x \cap A_i) \geq i\}$$

and call it the closed Schubert-cell of the sequence \underline{A} . In particular if

$$0 < \gamma_1 < \gamma_2 < \dots < \gamma_n \leq n+m$$

is a strictly increasing sequence of natural numbers less than or equal to $n+m$ then we define (setting $\gamma = (\gamma_1, \dots, \gamma_n)$)

$$(4.3) \quad SC(\gamma) = SC(\mathbf{C}^{\gamma_1}, \dots, \mathbf{C}^{\gamma_n})$$

where \mathbf{C}^r is viewed as the subspace of all vectors in \mathbf{C}^{n+m} whose last $n+m-r$ coordinates are zero.

4.4 *Flag Manifolds and the Bruhat Decomposition.* A flag in \mathbf{C}^{n+m} is a sequence of subspaces $\underline{F} = F_1 \subset \dots \subset F_{n+m} \subset \mathbf{C}^{n+m}$ such that $\dim F_i = i$. Let $Fl(\mathbf{C}^{n+m})$ denote the analytic manifold of all flags in \mathbf{C}^{n+m} . There is a natural holomorphic mapping $Fl(\mathbf{C}^{n+m}) \rightarrow G_n(\mathbf{C}^{n+m})$ given by associating to a flag \underline{F} its n -th element F_n . The flag manifold can be seen as the space of all cosets Bg , $g \in GL_{n+m}(\mathbf{C})$ where B is the Borel subgroup of all lower triangular matrices

in $\mathbf{GL}_{n+m}(\mathbf{C})$. The mapping $\mathbf{GL}_{n+m}(\mathbf{C}) \rightarrow Fl(\mathbf{C}^{n+m})$ associates to a matrix g the flag $\underline{F}(g)$ whose i -th element is the subspace of \mathbf{C}^{n+m} spanned by the first i row vectors of g .

Now view S_{n+m} , the symmetric group on $n+m$ letters as a subgroup of $\mathbf{GL}_{n+m}(\mathbf{C})$ by letting it permute the basis vectors ($\sigma(e_i) = e_{\sigma(i)}$). Then in $\mathbf{GL}_{n+m}(\mathbf{C})$ we have the so-called Bruhat decomposition.

$$(4.5) \quad \mathbf{GL}_{n+m}(\mathbf{C}) = \bigcup_{\sigma} B \sigma B \quad (\text{disjoint union})$$

Where σ runs through the Weyl group S_{n+m} of $\mathbf{GL}_{n+m}(\mathbf{C})$. An analogous decomposition holds in a considerably more general setting (reductive groups, cf. [24], section 28).

4.6. *The Bruhat order (also sometimes called Bernstein-Gelfand-Gelfand, or BGG order).* The closure of a double coset $B \sigma B$ is necessarily a union of other double cosets (by continuity). This defines an ordering on the Weyl group S_{n+m} defined by

$$(4.7) \quad \sigma > \tau \leftrightarrow \overline{B \sigma B} \supset B \tau B$$

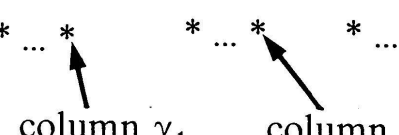
This ordering plays a considerable role in the study of cohomology of flag spaces [1] and also in the theory of highest weight representations [25, 26].

Let H be the subgroup of $\mathbf{G}_{n+m}(\mathbf{C})$ consisting of all block lower triangular matrices of the form $\begin{pmatrix} S_{11} & 0 \\ S_{21} & S_{22} \end{pmatrix}$, $S_{11} \in \mathbf{GL}_n(\mathbf{C})$, $S_{22} \in \mathbf{GL}_m(\mathbf{C})$, S_{21} an arbitrary $m \times n$ matrix. Then, using the remarks made in subsection 4.4 above, one sees that $\mathbf{G}_n(\mathbf{C}^{n+m})$ is the coset space $\{Hg \mid g \in \mathbf{GL}_{n+m}(\mathbf{C})\}$. Now let $\sigma \in S_{n+m}$ and let $\gamma_1 < \dots < \gamma_n$ be the n natural numbers in increasing order determined by

$$\sigma(e_{\gamma_i}) \in \{e_1, \dots, e_n\}, i = 1, \dots, n.$$

Then one easily sees that the image of $B \sigma B$ under $\mathbf{GL}_{n+m}(\mathbf{C}) \rightarrow \mathbf{G}_n(\mathbf{C}^{n+m})$, i.e. the set of all spaces spanned by matrices of the form $h \sigma b$, $h \in H$, $b \in B$, is the open Schubert cell of all elements in $\mathbf{G}_n(\mathbf{C}^{n+m})$ spanned by the rows of a matrix of the form

$$\begin{array}{cccccc} * & \dots & * & 0 & \dots & 0 & 0 & \dots & 0 \\ * & \dots & * & * & \dots & * & 0 & \dots & 0 \\ * & \dots & * & * & \dots & * & \dots & * & 0 \dots 0 \end{array}$$



where the last * in each row is nonzero. The closure of this open Schubert-cell is the Schubert-cell $SC(\gamma)$ defined in (4.3) above.

One easily checks that

$$(4.8) \quad SC(\mu) \subset SC(\gamma) \leftrightarrow \mu_i \leq \gamma_i, i = 1, \dots, n$$

and this order on the Schubert cells $SC(\gamma)$, or the equivalent ordering on n -tuples of natural numbers, is therefore a quotient of the Bruhat order on the Weyl group S_{n+m} . It is the induced order on the set of cosets $(S_n \times S_m)\sigma$, $\sigma \in S_{n+m}$. (Obviously if $\tau \in S_n \times S_m$, then $\tau\sigma(e_{\gamma_i}) \in \{e_1, \dots, e_n\}$ if $\sigma(e_{\gamma_i}) \in \{e_1, \dots, e_n\}$.) (And inversely the Bruhat order is determined by the associated orders of Schubert cells in the sense that $\sigma > \tau$ in S_n iff for all $k = 1, \dots, n - 1$ we have for the associated Schubert cells in $G_k(\mathbb{C}^n)$ that $SC(\sigma) \supset SC(\tau)$; this is a rather efficient way of calculating the Bruhat order on the Weyl group S_n .)

5. INTERRELATIONS

Now that we have defined the concepts we need we can start to describe some interrelations between the various manifestations of the specialization order we discussed in section 2 above.

5.1. *Overview of the Various Relations.* A schematic overview of the various interconnections is given by the following diagram. In this diagram we

