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REPRESENTATIONS OF THE SYMMETRIC GROUP, THE SPECIALIZATION ORDER, SYSTEMS AND GRASSMANN MANIFOLDS [1]

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ABSTRACT

A certain partial order on the set of all partitions of a given natural number n describes many containment, specialization or degeneration relations in the, seemingly, rather disparate parts of mathematics dealing with permutation representations of S_n, the existence of (0, 1)-matrices with prescribed row and column sums, symmetric mean inequalities, orbits of nilpotent matrices under similarity, Kronecker indices of control systems, doubly stochastic matrices and vectorbundles over the Riemann sphere. In this paper we discuss relations between all these subjects which show why the same ordering *must* appear all the time. Central to the discussion is the Schubert-cell decomposition of a Grassmann manifold and the associated (closure) ordering which is a quotient of the Bruhat ordering on the Weyl group S_n.

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1. Introduction

Let κ be a partition of n, $\kappa = (\kappa_1, ..., \kappa_m)$, $\kappa_1 \geqslant \cdots \geqslant \kappa_m \geqslant 0$, $\Sigma\kappa_i = n$. We identify partitions $(\kappa_1, ..., \kappa_m)$ and $(\kappa_1, ..., \kappa_m, 0, ..., 0)$. Quite a few classes of objects in mathematics are of course classified by partitions and often inclusion, specialization or degeneration relations between these objects are described by a certain partial order on the set of partitions. This partial order on the set of all partitions of n is defined as follows:

$$(1.1) \qquad (\kappa_1, ..., \kappa_m) > (\kappa_1', ..., \kappa_m') \quad \text{iff} \quad \sum_{i=1}^{r} \kappa_i \leqslant \sum_{i=1}^{r} \kappa_i', \quad r = 1, ..., m\,.$$

Thus, for example $(2, 2, 1) > (3, 2)$. If $\kappa > \kappa'$ we say that κ specializes to κ' or that κ is more general than κ'. The reverse order has been variously called the dominance order [2], the Snapper order [34, 41] or the natural order [35]. It occurs naturally in several seemingly rather unrelated parts of mathematics. Some of these occurrences are the

(i) Snapper, Liebler-Vitale, Lam, Young theorem (on the permutation representations of the symmetric groups)

(ii) Gale-Ryser theorem (on existence of (0, 1)-matrices)

(iii) Muirhead's inequality (a symmetric mean inequality)

(iv) Gerstenhaber-Hesselink theorem (on orbit closure properties of SL_n acting on nilpotent matrices)

(v) Kronecker indices (on the orbit closure, or degeneration, properties of linear control systems acted on by the socalled feedback group)