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EXACT SEQUENCES OF WITT GROUPS OF EQUIVARIANT FORMS

by D. W. Lewis

We construct two exact octagons i.e. circular eight-term exact sequences of Witt groups of forms invariant under the action of a finite group. When the group is trivial our octagons reduce to the two exact sequences obtained in [3]. See also [4].

We are indebted to Cl. Cibils and M. Kervaire for their suggestions to improve the original version of this paper.

Let F be a skewfield, J an involution on F i.e. an anti-automorphism of period two. We allow the case of J being the identity if F is commutative. Let π be a finite group.

Definition. A *form over* (F, π, J) is a map $\phi : V \times V \to F$, V an $F\pi$-module finite dimensional over F, which is sesquilinear, hermitian symmetric with respect to J, and π-invariant in that $\phi(gx, gy) = \phi(x, y)$ for all $g \in \pi$, all $x, y \in V$. Our forms are assumed to be non-singular i.e. $V \to V^*$, $x \to \phi(x, -)$ is bijective for all $x \in V$, where V^* is the F-dual of V. We write $W(F, \pi, J)$ for the Witt group of non-singular forms over (F, π, J), our definition of Witt group being as in [1]. (Remark—the forms which have Witt class zero are precisely those which are neutral i.e. which contain a submodule equal to its orthogonal complement. Note that we do not insist that this submodule be a direct summand as is required in another definition of Witt group which occurs in the literature. When char F does not divide $|\pi|$ then there is of course no difference between the two definitions of Witt group but in general they will be different.)

Now let K be a field, char $K \neq 2$, and let L be a quadratic extension of K so that $L = K(i)$, $i^2 = a$ for some $a \in K$. L admits both the identity map and the map—given by $\bar{i} = -i$ as involutions. We will consider the groups $W(K, \pi, 1)$, $W(L, \pi, 1)$ and $W(L, \pi, -)$. Also we write $W_{-1}(K, \pi, 1)$, $W_{-1}(L, \pi, 1)$ for the Witt groups of non-singular forms ϕ defined as above except that now ϕ is required to be skew-symmetric i.e. $\phi(y, x) = -\phi(x, y)$ for all $x, y \in V$. Also we write $W_{-1}(L, \pi, -)$ for the Witt group of skew-hermitian forms over L, i.e. $\phi(y, x) = -\overline{\phi(x, y)}$ for all $x, y \in V$. Note that for $\pi = 1$, the groups

$W_{-1}(K, \pi, 1)$, $W_{-1}(L, \pi, 1)$ are trivial since the skew-symmetric forms are even-dimensional and classified by rank alone [2, p. 334]. Note also that $W_{-1}(L, \pi, -)$ is isomorphic to $W(L, \pi, -)$ because if ϕ is hermitian then $i\phi$ is skew-hermitian and vice versa.

Let the trace maps $T_\alpha : L \to K$, $\alpha = 1, 2$ be defined by

$$T_\alpha(r_1 + r_2 i) = r_\alpha, \alpha = 1, 2$$

where each $r_\alpha \in K$. These trace maps induce in an obvious way maps between Witt groups as follows:

$$W(L, \pi, -) \xrightarrow{T_1} W(K, \pi, 1),$$

$$W(L, \pi, 1) \xrightarrow{T_2} W(K, \pi, 1),$$

$$W_{-1}(L, \pi, -) \xrightarrow{T'_1} W_{-1}(K, \pi, 1),$$

$$W_{-1}(L, \pi, 1) \xrightarrow{T'_2} W_{-1}(K, \pi, 1).$$

We denote the last two maps by T'_1, T'_2 merely to distinguish them from the first two maps.

Also we may use the tensor product in a natural way to define maps

$$U_1 : W(K, \pi, 1) \to W(L, \pi, 1)$$
$$U'_1 : W_{-1}(K, \pi, 1) \to W_{-1}(L, \pi, 1)$$

and there are also maps

$$U_2 : W(K, \pi, 1) \to W_{-1}(L, \pi, -)$$
$$U'_2 : W_{-1}(K, \pi, 1) \to W(L, \pi, -)$$

given by tensor product together with multiplication by the element $i \in L$. E.g. given a form $\phi : V \times V \to K$ over $(K, \pi, 1)$, $U_2(\phi)$ is the map

$$V \otimes_K L \times V \otimes_K L \to L$$

given by

$$(U_2(\phi))\,(x \otimes \lambda, y \otimes \mu) = \bar{\lambda} i \phi(x, y)\mu$$

for all $x, y \in V$, all $\lambda, \mu \in L$. It is easily checked that all these maps are well-defined.

THEOREM 1. *There is an exact octagon of Witt groups*

$$\begin{tikzcd}
 & W(L, \pi, -) \arrow[r, "T_1"] & W(K, \pi, 1) \arrow[dr, "U_1"] & \\
W_{-1}(K, \pi, 1) \arrow[ur, "U'_2"] & & & W(L, \pi, 1) \arrow[d, "T_2"] \\
W_{-1}(L, \pi, 1) \arrow[u, "T'_2"] & & & W(K, \pi, 1) \arrow[dl, "U_2"] \\
 & W_{-1}(K, \pi, 1) \arrow[ul, "U'_1"] & W_{-1}(L, \pi, -) \arrow[l, "T'_1"] &
\end{tikzcd}$$

Proof. We first show exactness of the portion

$$W(L, \pi, -) \xrightarrow{T_1} W(K, \pi, 1) \xrightarrow{U_1} W(L, \pi, 1),$$

i.e. we show that image of T_1 is the kernel of U_1.

Let $\phi : V \times V \to L$ represent an element of $W(L, \pi, -)$. To see that $U_1 T_1 \phi$ is neutral as a form over $(L, \pi, 1)$ we consider the subspace W of $V \otimes_K L$ as defined by

$$W = \{iv \otimes 1 + v \otimes i : v \in V\}.$$

Clearly W is an $L\pi$-submodule and $2 \dim_K W = \dim_K(V \otimes_K L)$. We will show that $W = W^{\perp}$, orthogonal complement with respect to $U_1 T_1 \phi$. Now if $v, v' \in V$ then

$$(U_1 T_1 \phi)\,(iv \otimes 1 + v \otimes i,\ iv' \otimes 1 + v' \otimes i)$$

is easily verified to be zero using the sesquilinearity of ϕ and the definitions of T_1, U_1. Thus $W \subset W^{\perp}$. It follows that in fact $W = W^{\perp}$ since they have the same dimension.

Next let $\psi : V \times V \to K$ represent an element of $W(K, \pi, 1)$. We may assume ψ is anisotropic by [1]. Now if $U_1 \psi$ is zero in $W(L, \pi, 1)$ then $V \otimes_K L$ contains a self-orthogonal L-submodule W. This enables us to define an L-space structure on V as follows:

Observe that

$$2 \dim_L W = \dim_L V \otimes_K L, \dim_L W = \dim_L V \otimes i,$$

and that $W \cap (V \otimes i) = 0$ since ψ is anisotropic. Thus $V \otimes_K L \cong (V \otimes i) \oplus W$. It now follows that, given $v \in V$, there exists a unique element $v' \in V$ such that $v \otimes 1 + v' \otimes i \in W$. Then define the operator $J : V \to V$ by $J(v') = v$ for each $v \in V$. It is easily verified that J is skew-adjoint, $J^2 = a$ and that J commutes with the π-action. Thus J can be used to give V an $L\pi$-module structure, $i \in L$ operating as J on V.

Now define a form $\phi : V \times V \to L$ by

$$\phi(x, y) = \psi(x, y) + i^{-1} \psi(x, Jy)$$

for all $x, y \in V$. Then ϕ is a non-singular form over $(L, \pi, -)$ and $T_1\phi = \psi$.

This proves exactness at $W(K, \pi, 1)$. At the three points in the sequence

$$W(L, \pi, 1) \xrightarrow{T2} W(K, \pi, 1) \xrightarrow{U2} W_{-1}(L, \pi, -),$$

$$W_{-1}(L, \pi, -) \xrightarrow{T'_1} W_{-1}(K, \pi, 1) \xrightarrow{U'_1} W_{-1}(L, \pi, 1),$$

$$W_{-1}(L, \pi, 1) \xrightarrow{T'_2} W_{-1}(K, \pi, 1) \xrightarrow{U'_2} W(L, \pi, -)$$

exactness is proven by the same arguments.

Now consider the piece

$$W_{-1}(K, \pi, 1) \xrightarrow{U'_1} W_{-1}(L, \pi, 1) \xrightarrow{T'_2} W_{-1}(K, \pi, 1).$$

If $\phi : V \times V \to K$ represents an element of $W_{-1}(K, \pi, 1)$ then we see that $T'_2U'_2\phi$ is neutral by looking at

$$W \subset V \otimes_K L, W = V \otimes 1$$

and checking that $W = W^{\perp}$.

$$T'_2U'_2\phi(v_1 \otimes 1, v' \otimes 1) = T'_2\phi(v, v') = 0$$

for all $v, v' \in V$ so that $W \subset W^{\perp}$. Hence $W = W^{\perp}$ since

$$2 \dim_K W = \dim_K V \otimes_K L.$$

Conversely if ψ, representing an element of $W_{-1}(L, \pi, 1)$, is such that $T'_2\psi$ is neutral then $\psi : V \times V \to L$, V an $L\pi$-module, and there exists a $K\pi$-module W of V with W $= W^{\perp}$, orthogonal complement with respect to $T'_2\psi$. Also

$2 \dim_K W = \dim_K V$. Defining $\phi : W \times W \to$ by $(x, y) = \psi(x, y)$ for $x, y \in W$ then $W \otimes_K L \cong V$ as $L\pi$-modules via the isomorphism

$$w \otimes \lambda \to \lambda w, \lambda \in L, w \in W.$$

Moreover $U'_1(\phi) = \psi$ completing the proof of exactness at $W_{-1}(L, \pi, 1)$.

For the three remaining points of the sequence, which each have U followed by T, the above arguments go through virtually unchanged.

This completes the proof.

Now suppose we have a quaternion division algebra D over K, $D = \left(\frac{a, b}{K}\right)$ generated by i, j with $i^2 = a, j^2 = b, ij = -ji$ etc. We have involutions $-$ and $\wedge$ on D given by $\bar{i} = -i, \bar{j} = -j$ and $\hat{i} = i, \hat{j} = j$ respectively. Let L be the maximal subfield $K(i)$ of D. There are trace maps $T_i : D \to L, i = 1, 2$ given by $T_i(z_1 + z_2 j) = z_1$ where $z_1, z_2 \in L$, and these induce natural maps of Witt groups

$$W(D, \pi, -) \xrightarrow{T_1} W(L, \pi, -),$$

$$W(D, \pi, \wedge) \xrightarrow{T_2} W(L, \pi, 1),$$

$$W(D, \pi, \wedge) \xrightarrow{T'_1} W(L, \pi, -),$$

$$W(D, \pi, -) \xrightarrow{T'_2} W_{-1}(L, \pi, 1).$$

Also we have maps

$$W(L, \pi, -) \xrightarrow{U_1} W(D, \pi, \wedge),$$

$$W(L, \pi, 1) \xrightarrow{U_2} W(D, \pi, \wedge),$$

$$W(L, \pi, -) \xrightarrow{U'_1} W(D, \pi, -),$$

$$W_{-1}(L, \pi, 1) \xrightarrow{U'_2} W(D, \pi, -),$$

U_1, U'_1 given by the tensor product, U_2, U'_2 by the tensor product together with multiplication by the element $k = ij$ of D. E.g. given a form $\phi : V \times V \to L$ over $(L, \pi, 1)$, $U_2(\phi)$ is the form $V \otimes_L D \times V \otimes_L D \to D$ defined by

$$U_2(\phi)(x \otimes \lambda, y \otimes \mu) = \hat{\lambda}\, \phi(x, y) k \mu \quad \text{for} \quad \lambda, \mu \in D, x, y \in V.$$

(Beware that the position of k matters as D is not commutative!).

THEOREM 2. *There is an exact octagon of Witt groups*

$$\begin{tikzcd}
 & W(D, \pi, -) \arrow[r, "T_1"] & W(L, \pi, -) \arrow[dr, "U_1"] & \\
W_{-1}(L, \pi, 1) \arrow[ur, "U'_2"] & & & W(D, \pi, \wedge) \arrow[d, "T_2"] \\
W(D, \pi, -) \arrow[u, "T'_2"] & & & W(L, \pi, 1) \arrow[dl, "U_2"] \\
 & W(L, \pi, -) \arrow[ul, "U'_1"] & W(D, \pi, \wedge) \arrow[l, "T'_1"] &
\end{tikzcd}$$

Proof. We need only modify the proof of theorem 1 slightly. Specifically j will play the role that i did in theorem 1. For example at the start of the proof we must put

$$W = (jv \otimes 1 + v \otimes j : v \in V\}$$

and later on the operator J is defined in a similar fashion to that of theorem 1 except that we get $J^2 = b$ leading to a $D\pi$-module structure. The lack of commutativity of D causes no problem, although care must be taken in dealing with the maps U_2, U'_2. (See the comment above.) We leave the reader to check that with these modifications the proof goes through completely.

Comment 1. When $\pi = 1$ the Witt groups $W_{-1}(K, \pi, 1)$ and $W_{-1}(L, \pi, 1)$ are trivial as we remarked earlier in this paper. Our sequences now reduce to those of [3].

Comment 2. Note that $W_{-1}(L, \pi, -) \cong W(L, \pi, -)$ for the reason stated earlier.

Also $W(D, \pi, \wedge) \cong W_{-1}(D, \pi, -)$ since forms hermitian with respect to $\wedge$ are equivalent to those skew-hermitian with respect to $-$ and vice versa. (The correspondence $\phi \leftrightarrow i\phi$ gives this since $\hat{x} = i^{-1}\bar{x}i$ for all $x \in D$.) A consequence of the above is that the two octagons each display an interesting symmetry feature. In the "antipodal" position to $W(F, \pi, J)$ in the octagon we always have $W_{-1}(F, \pi, J)$.

Comment 3. Our proof is different from that of [3] and it may well be possible that this new method of proof can also be used to generalize the sequences of [3] to the case when K is a commutative ring and L is some kind of Galois extension with Galois group cyclic of order two.

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