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EXACT SEQUENCES OF WITT GROUPS OF EQUIVARIANT FORMS

by D. W. LEWIS

We construct two exact octagons i.e. circular eight-term exact sequences of Witt groups of forms invariant under the action of a finite group. When the group is trivial our octagons reduce to the two exact sequences obtained in [3]. See also [4].

We are indebted to Cl. Cibils and M. Kervaire for their suggestions to improve the original version of this paper.

Let F be a skewfield, J an involution on F i.e. an anti-automorphism of period two. We allow the case of J being the identity if F is commutative. Let π be a finite group.

Definition. A form over (F, π, J) is a map $\phi: V \times V \to F$, V an $F\pi$ -module finite dimensional over F, which is sesquilinear, hermitian symmetric with respect to J, and π -invariant in that $\phi(gx, gy) = \phi(x, y)$ for all $g \in \pi$, all $x, y \in V$. Our forms are assumed to be non-singular i.e. $V \to V^*, x \to \phi(x, -)$ is bijective for all $x \in V$, where V^* is the F-dual of V. We write $W(F, \pi, J)$ for the Witt group of non-singular forms over (F, π, J) , our definition of Witt group being as in [1]. (Remark—the forms which have Witt class zero are precisely those which are neutral i.e. which contain a submodule equal to its orthogonal complement. Note that we do not insist that this submodule be a direct summand as is required in another definition of Witt group which occurs in the literature. When char F does not divide $|\pi|$ then there is of course no difference between the two definitions of Witt group but in general they will be different.)

Now let K be a field, char $K \neq 2$, and let L be a quadratic extension of K so that L = K(i), $i^2 = a$ for some $a \in K$. L admits both the identity map and the map—given by $\bar{\imath} = -i$ as involutions. We will consider the groups $W(K, \pi, 1)$, $W(L, \pi, 1)$ and $W(L, \pi, -)$. Also we write $W_{-1}(K, \pi, 1)$, $W_{-1}(L, \pi, 1)$ for the Witt groups of non-singular forms ϕ defined as above except that now ϕ is required to be skew-symmetric i.e. $\phi(y, x) = -\phi(x, y)$ for all $x, y \in V$. Also we write $W_{-1}(L, \pi, -)$ for the Witt group of skew-hermitian forms over L, i.e. $\phi(y, x) = -\phi(x, y)$ for all $x, y \in V$. Note that for $\pi = 1$, the groups

 $W_{-1}(K, \pi, 1)$, $W_{-1}(L, \pi, 1)$ are trivial since the skew-symmetric forms are even-dimensional and classified by rank alone [2, p. 334]. Note also that $W_{-1}(L, \pi, -)$ is isomorphic to $W(L, \pi, -)$ because if ϕ is hermitian then $i\phi$ is skew-hermitian and vice versa.

Let the trace maps $T_{\alpha}: L \to K$, $\alpha = 1, 2$ be defined by

$$T_{\alpha}(r_1 + r_2 i) = r_{\alpha}, \alpha = 1, 2$$

where each $r_{\alpha} \in K$. These trace maps induce in an obvious way maps between Witt groups as follows:

$$W(L, \pi, -) \xrightarrow{T_1} W(K, \pi, 1),$$

$$W(L, \pi, 1) \xrightarrow{T_2} W(K, \pi, 1),$$

$$W_{-1}(L, \pi, -) \xrightarrow{T'_1} W_{-1}(K, \pi, 1),$$

$$W_{-1}(L, \pi, 1) \xrightarrow{T'_2} W_{-1}(K, \pi, 1).$$

We denote the last two maps by T'_1 , T'_2 merely to distinguish them from the first two maps.

Also we may use the tensor product in a natural way to define maps

$$U_1: W(K, \pi, 1) \to W(L, \pi, 1)$$

 $U'_1: W_{-1}(K, \pi, 1) \to W_{-1}(L, \pi, 1)$

and there are also maps

$$U_2: W(K, \pi, 1) \to W_{-1}(L, \pi, -)$$

 $U_2: W_{-1}(K, \pi, 1) \to W(L, \pi, -)$

given by tensor product together with multiplication by the element $i \in L$. E.g. given a form $\phi: V \times V \to K$ over $(K, \pi, 1)$, $U_2(\phi)$ is the map

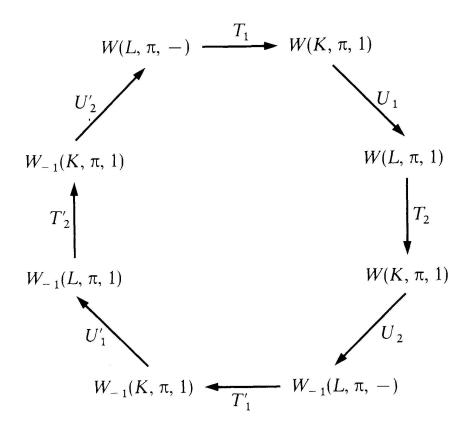
$$V \otimes_K L \times V \otimes_K L \to L$$

given by

$$(U_2(\phi))(x \otimes \lambda, y \otimes \mu) = \overline{\lambda} i \phi(x, y) \mu$$

for all $x, y \in V$, all $\lambda, \mu \in L$. It is easily checked that all these maps are well-defined.

THEOREM 1. There is an exact octagon of Witt groups



Proof. We first show exactness of the portion

$$W(L, \pi, -) \stackrel{T_1}{\rightarrow} W(K, \pi, 1) \stackrel{U_1}{\rightarrow} W(L, \pi, 1)$$

i.e. we show that image of T_1 is the kernel of U_1 .

Let $\phi: V \times V \to L$ represent an element of $W(L, \pi, -)$. To see that $U_1 T_1 \phi$ is neutral as a form over $(L, \pi, 1)$ we consider the subspace W of $V \otimes_K L$ as defined by

$$W = \{iv \otimes 1 + v \otimes i : v \in V\}.$$

Clearly W is an $L\pi$ -submodule and $2 \dim_K W = \dim_K (V \otimes_K L)$. We will show that $W = W^{\perp}$, orthogonal complement with respect to $U_1 T_1 \varphi$. Now if $v, v' \in V$ then

$$(U_1 T_1 \phi) (iv \otimes 1 + v \otimes i, iv' \otimes 1 + v' \otimes i)$$

is easily verified to be zero using the sesquilinearity of ϕ and the definitions of T_1 , U_1 . Thus $W \subset W^{\perp}$. It follows that in fact $W = W^{\perp}$ since they have the same dimension.

Next let $\psi: V \times V \to K$ represent an element of $W(K, \pi, 1)$. We may assume ψ is anisotropic by [1]. Now if $U_1\psi$ is zero in $W(L, \pi, 1)$ then $V \otimes_K L$ contains a self-orthogonal L-submodule W. This enables us to define an L-space structure on V as follows:

Observe that

$$2 \dim_L W = \dim_L V \otimes_K L, \dim_L W = \dim_L V \otimes i,$$

and that $W \cap (V \otimes i) = 0$ since ψ is anisotropic. Thus $V \otimes_K L \cong (V \otimes i) \oplus W$. It now follows that, given $v \in V$, there exists a unique element $v' \in V$ such that $v \otimes 1 + v' \otimes i \in W$. Then define the operator $J: V \to V$ by J(v') = v for each $v \in V$. It is easily verified that J is skew-adjoint, $J^2 = a$ and that J commutes with the π -action. Thus J can be used to give V an $L\pi$ -module structure, $i \in L$ operating as J on V.

Now define a form $\phi: V \times V \to L$ by

$$\phi(x, y) = \psi(x, y) + i^{-1} \psi(x, Jy)$$

for all $x, y \in V$. Then ϕ is a non-singular form over $(L, \pi, -)$ and $T_1 \phi = \psi$. This proves exactness at $W(K, \pi, 1)$. At the three points in the sequence

$$W(L, \pi, 1) \xrightarrow{T_2} W(K, \pi, 1) \xrightarrow{U_2} W_{-1}(L, \pi, -),$$

$$W_{-1}(L, \pi, -) \xrightarrow{T'_1} W_{-1}(K, \pi, 1) \xrightarrow{U'_1} W_{-1}(L, \pi, 1),$$

$$W_{-1}(L, \pi, 1) \xrightarrow{T'_2} W_{-1}(K, \pi, 1) \xrightarrow{U'_2} W(L, \pi, -)$$

exactness is proven by the same arguments.

Now consider the piece

$$W_{-1}(K, \pi, 1) \stackrel{U'_1}{\to} W_{-1}(L, \pi, 1) \stackrel{T'_2}{\to} W_{-1}(K, \pi, 1).$$

If $\phi: V \times V \to K$ represents an element of $W_{-1}(K, \pi, 1)$ then we see that $T_2'U_2'\phi$ is neutral by looking at

$$W \subset V \otimes_K L, W = V \otimes 1$$

and checking that $W = W^{\perp}$.

$$T_2'U_2'\phi(v_1\otimes 1, v'\otimes 1) = T_2'\phi(v, v') = 0$$

for all $v, v' \in V$ so that $W \subset W^{\perp}$. Hence $W = W^{\perp}$ since

$$2 \dim_K W = \dim_K V \otimes_K L.$$

Conversely if ψ , representing an element of $W_{-1}(L, \pi, 1)$, is such that $T_2'\psi$ is neutral then $\psi: V \times V \to L$, V an $L\pi$ -module, and there exists a $K\pi$ -module W of V with $W = W^{\perp}$, orthogonal complement with respect to $T_2'\psi$. Also

2 dim_K $W = \dim_K V$. Defining $\phi : W \times W \to \text{ by } (x, y) = \psi(x, y)$ for $x, y \in W$ then $W \otimes_K L \cong V$ as $L\pi$ -modules via the isomorphism

$$w \otimes \lambda \rightarrow \lambda w, \lambda \in L, w \in W.$$

Moreover $U'_1(\phi) = \psi$ completing the proof of exactness at $W_{-1}(L, \pi, 1)$. For the three remaining points of the sequence, which each have U followed by T, the above arguments go through virtually unchanged.

This completes the proof.

Now suppose we have a quaternion division algebra D over K, $D = \left(\frac{a, b}{K}\right)$ generated by i, j with $i^2 = a, j^2 = b$, ij = -ji etc. We have involutions — and \hat{i} on D given by $\bar{i} = -i, \bar{j} = -j$ and $\hat{i} = i, \hat{j} = j$ respectively. Let L be the maximal subfield K(i) of D. There are trace maps $T_i: D \to L$, i = 1, 2 given by $T_i(z_1 + z_2 j) = z_1$ where $z_1, z_2 \in L$, and these induce natural maps of Witt groups

$$W(D, \pi, -) \xrightarrow{T_1} W(L, \pi, -),$$

$$W(D, \pi, ^) \xrightarrow{T_2} W(L, \pi, 1),$$

$$W(D, \pi, ^) \xrightarrow{T_1'} W(L, \pi, -),$$

$$W(D, \pi, -) \xrightarrow{T_2'} W_{-1}(L, \pi, 1).$$

Also we have maps

$$W(L, \pi, -) \xrightarrow{U_1} W(D, \pi, ^),$$

$$W(L, \pi, 1) \xrightarrow{U_2} W(D, \pi, ^),$$

$$W(L, \pi, -) \xrightarrow{U'_1} W(D, \pi, -),$$

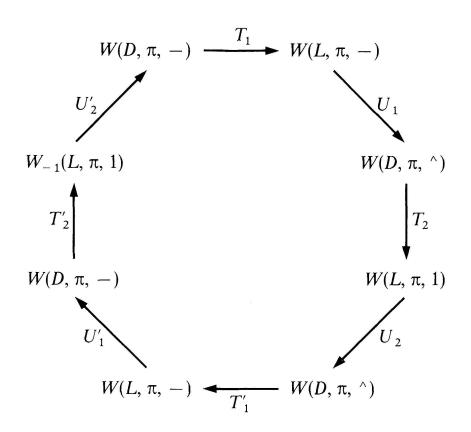
$$W_{-1}(L, \pi, 1) \xrightarrow{U'_2} W(D, \pi, -),$$

 U_1 , U_1' given by the tensor product, U_2 , U_2' by the tensor product together with multiplication by the element k = ij of D. E.g. given a form $\phi: V \times V \to L$ over $(L, \pi, 1)$, $U_2(\phi)$ is the form $V \otimes_L D \times V \otimes_L D \to D$ defined by

$$U_2(\varphi)\left(x\otimes\lambda,\,y\otimes\mu\right)\,=\,\widehat{\lambda}\,\,\varphi(x,\,y)k\mu\quad\text{for}\quad\lambda,\,\mu\in D,\,x,\,y\in V.$$

(Beware that the position of k matters as D is not commutative!).

THEOREM 2. There is an exact octagon of Witt groups



Proof. We need only modify the proof of theorem 1 slightly. Specifically j will play the role that i did in theorem 1. For example at the start of the proof we must put

$$W = \{jv \otimes 1 + v \otimes j : v \in V\}$$

and later on the operator J is defined in a similar fashion to that of theorem 1 except that we get $J^2 = b$ leading to a $D\pi$ -module structure. The lack of commutativity of D causes no problem, although care must be taken in dealing with the maps U_2 , U_2' . (See the comment above.) We leave the reader to check that with these modifications the proof goes through completely.

Comment 1. When $\pi = 1$ the Witt groups $W_{-1}(K, \pi, 1)$ and $W_{-1}(L, \pi, 1)$ are trivial as we remarked earlier in this paper. Our sequences now reduce to those of [3].

Comment 2. Note that $W_{-1}(L, \pi, -) \cong W(L, \pi, -)$ for the reason stated earlier.

Also $W(D, \pi, ^{\wedge}) \cong W_{-1}(D, \pi, -)$ since forms hermitian with respect to $^{\wedge}$ are equivalent to those skew-hermitian with respect to - and vice versa. (The correspondence $\phi \leftrightarrow i\phi$ gives this since $\hat{x} = i^{-1}\bar{x}i$ for all $x \in D$.) A consequence of the above is that the two octagons each display an interesting symmetry

feature. In the "antipodal" position to $W(F, \pi, J)$ in the octagon we always have $W_{-1}(F, \pi, J)$.

Comment 3. Our proof is different from that of [3] and it may well be possible that this new method of proof can also be used to generalize the sequences of [3] to the case when K is a commutative ring and L is some kind of Galois extension with Galois group cyclic of order two.

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