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## EXACT SEQUENCES OF WITT GROUPS OF EQUIVARIANT FORMS

by D. W. LEWIS

We construct two exact octagons i.e. circular eight-term exact sequences of Witt groups of forms invariant under the action of a finite group. When the group is trivial our octagons reduce to the two exact sequences obtained in [3]. See also [4].

We are indebted to Cl. Cibils and M. Kervaire for their suggestions to improve the original version of this paper.

Let  $F$  be a skewfield,  $J$  an involution on  $F$  i.e. an anti-automorphism of period two. We allow the case of  $J$  being the identity if  $F$  is commutative. Let  $\pi$  be a finite group.

*Definition.* A form over  $(F, \pi, J)$  is a map  $\phi : V \times V \rightarrow F$ ,  $V$  an  $F\pi$ -module finite dimensional over  $F$ , which is sesquilinear, hermitian symmetric with respect to  $J$ , and  $\pi$ -invariant in that  $\phi(gx, gy) = \phi(x, y)$  for all  $g \in \pi$ , all  $x, y \in V$ . Our forms are assumed to be non-singular i.e.  $V \rightarrow V^*$ ,  $x \rightarrow \phi(x, -)$  is bijective for all  $x \in V$ , where  $V^*$  is the  $F$ -dual of  $V$ . We write  $W(F, \pi, J)$  for the Witt group of non-singular forms over  $(F, \pi, J)$ , our definition of Witt group being as in [1]. (Remark—the forms which have Witt class zero are precisely those which are neutral i.e. which contain a submodule equal to its orthogonal complement. Note that we do not insist that this submodule be a direct summand as is required in another definition of Witt group which occurs in the literature. When  $\text{char } F$  does not divide  $|\pi|$  then there is of course no difference between the two definitions of Witt group but in general they will be different.)

Now let  $K$  be a field,  $\text{char } K \neq 2$ , and let  $L$  be a quadratic extension of  $K$  so that  $L = K(i)$ ,  $i^2 = a$  for some  $a \in K$ .  $L$  admits both the identity map and the map—given by  $\bar{i} = -i$  as involutions. We will consider the groups  $W(K, \pi, 1)$ ,  $W(L, \pi, 1)$  and  $W(L, \pi, -)$ . Also we write  $W_{-1}(K, \pi, 1)$ ,  $W_{-1}(L, \pi, 1)$  for the Witt groups of non-singular forms  $\phi$  defined as above except that now  $\phi$  is required to be skew-symmetric i.e.  $\phi(y, x) = -\phi(x, y)$  for all  $x, y \in V$ . Also we write  $W_{-1}(L, \pi, -)$  for the Witt group of skew-hermitian forms over  $L$ , i.e.  $\phi(y, x) = -\phi(x, y)$  for all  $x, y \in V$ . Note that for  $\pi = 1$ , the groups

$W_{-1}(K, \pi, 1)$ ,  $W_{-1}(L, \pi, 1)$  are trivial since the skew-symmetric forms are even-dimensional and classified by rank alone [2, p. 334]. Note also that  $W_{-1}(L, \pi, -)$  is isomorphic to  $W(L, \pi, -)$  because if  $\phi$  is hermitian then  $i\phi$  is skew-hermitian and vice versa.

Let the trace maps  $T_\alpha : L \rightarrow K$ ,  $\alpha = 1, 2$  be defined by

$$T_\alpha(r_1 + r_2i) = r_\alpha, \alpha = 1, 2$$

where each  $r_\alpha \in K$ . These trace maps induce in an obvious way maps between Witt groups as follows:

$$\begin{aligned} W(L, \pi, -) &\xrightarrow{T_1} W(K, \pi, 1), \\ W(L, \pi, 1) &\xrightarrow{T_2} W(K, \pi, 1), \\ W_{-1}(L, \pi, -) &\xrightarrow{T'_1} W_{-1}(K, \pi, 1), \\ W_{-1}(L, \pi, 1) &\xrightarrow{T'_2} W_{-1}(K, \pi, 1). \end{aligned}$$

We denote the last two maps by  $T'_1, T'_2$  merely to distinguish them from the first two maps.

Also we may use the tensor product in a natural way to define maps

$$\begin{aligned} U_1 : W(K, \pi, 1) &\rightarrow W(L, \pi, 1) \\ U'_1 : W_{-1}(K, \pi, 1) &\rightarrow W_{-1}(L, \pi, 1) \end{aligned}$$

and there are also maps

$$\begin{aligned} U_2 : W(K, \pi, 1) &\rightarrow W_{-1}(L, \pi, -) \\ U'_2 : W_{-1}(K, \pi, 1) &\rightarrow W(L, \pi, -) \end{aligned}$$

given by tensor product together with multiplication by the element  $i \in L$ . E.g. given a form  $\phi : V \times V \rightarrow K$  over  $(K, \pi, 1)$ ,  $U_2(\phi)$  is the map

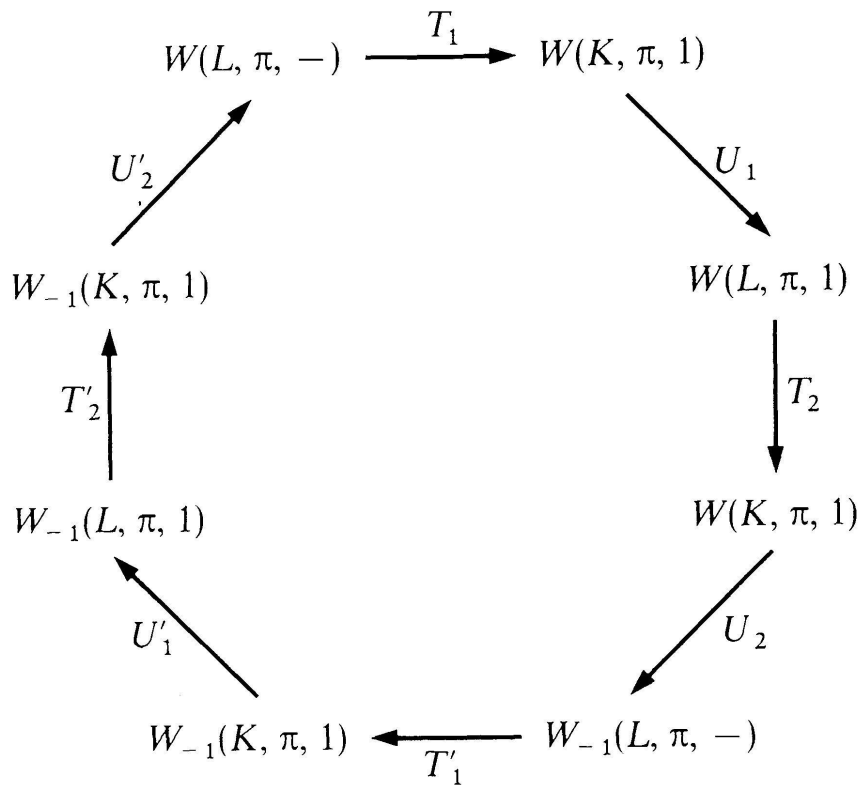
$$V \otimes_K L \times V \otimes_K L \rightarrow L$$

given by

$$(U_2(\phi))(x \otimes \lambda, y \otimes \mu) = \bar{\lambda}i\phi(x, y)\mu$$

for all  $x, y \in V$ , all  $\lambda, \mu \in L$ . It is easily checked that all these maps are well-defined.

THEOREM 1. *There is an exact octagon of Witt groups*



*Proof.* We first show exactness of the portion

$$W(L, \pi, -) \xrightarrow{T_1} W(K, \pi, 1) \xrightarrow{U_1} W(L, \pi, 1),$$

i.e. we show that image of  $T_1$  is the kernel of  $U_1$ .

Let  $\phi : V \times V \rightarrow L$  represent an element of  $W(L, \pi, -)$ . To see that  $U_1 T_1 \phi$  is neutral as a form over  $(L, \pi, 1)$  we consider the subspace  $W$  of  $V \otimes_K L$  as defined by

$$W = \{iv \otimes 1 + v \otimes i : v \in V\}.$$

Clearly  $W$  is an  $L\pi$ -submodule and  $2 \dim_K W = \dim_K(V \otimes_K L)$ . We will show that  $W = W^\perp$ , orthogonal complement with respect to  $U_1 T_1 \phi$ . Now if  $v, v' \in V$  then

$$(U_1 T_1 \phi)(iv \otimes 1 + v \otimes i, iv' \otimes 1 + v' \otimes i)$$

is easily verified to be zero using the sesquilinearity of  $\phi$  and the definitions of  $T_1, U_1$ . Thus  $W \subset W^\perp$ . It follows that in fact  $W = W^\perp$  since they have the same dimension.

Next let  $\psi : V \times V \rightarrow K$  represent an element of  $W(K, \pi, 1)$ . We may assume  $\psi$  is anisotropic by [1]. Now if  $U_1 \psi$  is zero in  $W(L, \pi, 1)$  then  $V \otimes_K L$  contains a self-orthogonal  $L$ -submodule  $W$ . This enables us to define an  $L$ -space structure on  $V$  as follows:

Observe that

$$2 \dim_L W = \dim_L V \otimes_K L, \dim_L W = \dim_L V \otimes i,$$

and that  $W \cap (V \otimes i) = 0$  since  $\psi$  is anisotropic. Thus  $V \otimes_K L \cong (V \otimes i) \oplus W$ . It now follows that, given  $v \in V$ , there exists a unique element  $v' \in V$  such that  $v \otimes 1 + v' \otimes i \in W$ . Then define the operator  $J: V \rightarrow V$  by  $J(v') = v$  for each  $v \in V$ . It is easily verified that  $J$  is skew-adjoint,  $J^2 = a$  and that  $J$  commutes with the  $\pi$ -action. Thus  $J$  can be used to give  $V$  an  $L\pi$ -module structure,  $i \in L$  operating as  $J$  on  $V$ .

Now define a form  $\phi: V \times V \rightarrow L$  by

$$\phi(x, y) = \psi(x, y) + i^{-1} \psi(x, Jy)$$

for all  $x, y \in V$ . Then  $\phi$  is a non-singular form over  $(L, \pi, -)$  and  $T_1\phi = \psi$ .

This proves exactness at  $W(K, \pi, 1)$ . At the three points in the sequence

$$W(L, \pi, 1) \xrightarrow{T_2} W(K, \pi, 1) \xrightarrow{U_2} W_{-1}(L, \pi, -),$$

$$W_{-1}(L, \pi, -) \xrightarrow{T'_1} W_{-1}(K, \pi, 1) \xrightarrow{U'_1} W_{-1}(L, \pi, 1),$$

$$W_{-1}(L, \pi, 1) \xrightarrow{T'_2} W_{-1}(K, \pi, 1) \xrightarrow{U'_2} W(L, \pi, -)$$

exactness is proven by the same arguments.

Now consider the piece

$$W_{-1}(K, \pi, 1) \xrightarrow{U'_1} W_{-1}(L, \pi, 1) \xrightarrow{T'_2} W_{-1}(K, \pi, 1).$$

If  $\phi: V \times V \rightarrow K$  represents an element of  $W_{-1}(K, \pi, 1)$  then we see that  $T'_2 U'_2 \phi$  is neutral by looking at

$$W \subset V \otimes_K L, W = V \otimes 1$$

and checking that  $W = W^\perp$ .

$$T'_2 U'_2 \phi(v_1 \otimes 1, v' \otimes 1) = T'_2 \phi(v, v') = 0$$

for all  $v, v' \in V$  so that  $W \subset W^\perp$ . Hence  $W = W^\perp$  since

$$2 \dim_K W = \dim_K V \otimes_K L.$$

Conversely if  $\psi$ , representing an element of  $W_{-1}(L, \pi, 1)$ , is such that  $T'_2 \psi$  is neutral then  $\psi: V \times V \rightarrow L$ ,  $V$  an  $L\pi$ -module, and there exists a  $K\pi$ -module  $W$  of  $V$  with  $W = W^\perp$ , orthogonal complement with respect to  $T'_2 \psi$ . Also

$2 \dim_K W = \dim_K V$ . Defining  $\phi : W \times W \rightarrow V$  by  $\phi(x, y) = \psi(x, y)$  for  $x, y \in W$  then  $W \otimes_K L \cong V$  as  $L\pi$ -modules via the isomorphism

$$w \otimes \lambda \rightarrow \lambda w, \lambda \in L, w \in W.$$

Moreover  $U'_1(\phi) = \psi$  completing the proof of exactness at  $W_{-1}(L, \pi, 1)$ .

For the three remaining points of the sequence, which each have  $U$  followed by  $T$ , the above arguments go through virtually unchanged.

This completes the proof.

Now suppose we have a quaternion division algebra  $D$  over  $K$ ,  $D = \left( \frac{a, b}{K} \right)$  generated by  $i, j$  with  $i^2 = a, j^2 = b, ij = -ji$  etc. We have involutions  $-$  and  $\hat{\phantom{x}}$  on  $D$  given by  $\bar{i} = -i, \bar{j} = -j$  and  $\hat{i} = i, \hat{j} = j$  respectively. Let  $L$  be the maximal subfield  $K(i)$  of  $D$ . There are trace maps  $T_i : D \rightarrow L, i = 1, 2$  given by  $T_i(z_1 + z_2 j) = z_1$  where  $z_1, z_2 \in L$ , and these induce natural maps of Witt groups

$$W(D, \pi, -) \xrightarrow{T_1} W(L, \pi, -),$$

$$W(D, \pi, \hat{\phantom{x}}) \xrightarrow{T_2} W(L, \pi, 1),$$

$$W(D, \pi, \hat{\phantom{x}}) \xrightarrow{T'_1} W(L, \pi, -),$$

$$W(D, \pi, -) \xrightarrow{T'_2} W_{-1}(L, \pi, 1).$$

Also we have maps

$$W(L, \pi, -) \xrightarrow{U_1} W(D, \pi, \hat{\phantom{x}}),$$

$$W(L, \pi, 1) \xrightarrow{U_2} W(D, \pi, \hat{\phantom{x}}),$$

$$W(L, \pi, -) \xrightarrow{U'_1} W(D, \pi, -),$$

$$W_{-1}(L, \pi, 1) \xrightarrow{U'_2} W(D, \pi, -),$$

$U_1, U'_1$  given by the tensor product,  $U_2, U'_2$  by the tensor product together with multiplication by the element  $k = ij$  of  $D$ . E.g. given a form  $\phi : V \times V \rightarrow L$  over  $(L, \pi, 1)$ ,  $U_2(\phi)$  is the form  $V \otimes_L D \times V \otimes_L D \rightarrow D$  defined by

$$U_2(\phi)(x \otimes \lambda, y \otimes \mu) = \hat{\lambda} \phi(x, y) k \mu \quad \text{for } \lambda, \mu \in D, x, y \in V.$$

(Beware that the position of  $k$  matters as  $D$  is not commutative!).

THEOREM 2. *There is an exact octagon of Witt groups*

$$\begin{array}{ccc}
 W(D, \pi, -) & \xrightarrow{T_1} & W(L, \pi, -) \\
 \nearrow U'_2 & & \searrow U_1 \\
 W_{-1}(L, \pi, 1) & & W(D, \pi, \wedge) \\
 \uparrow T'_2 & & \downarrow T_2 \\
 W(D, \pi, -) & & W(L, \pi, 1) \\
 \nwarrow U'_1 & & \swarrow U_2 \\
 W(L, \pi, -) & \xleftarrow{T'_1} & W(D, \pi, \wedge)
 \end{array}$$

*Proof.* We need only modify the proof of theorem 1 slightly. Specifically  $j$  will play the role that  $i$  did in theorem 1. For example at the start of the proof we must put

$$W = \{jv \otimes 1 + v \otimes j : v \in V\}$$

and later on the operator  $J$  is defined in a similar fashion to that of theorem 1 except that we get  $J^2 = b$  leading to a  $D\pi$ -module structure. The lack of commutativity of  $D$  causes no problem, although care must be taken in dealing with the maps  $U_2, U'_2$ . (See the comment above.) We leave the reader to check that with these modifications the proof goes through completely.

*Comment 1.* When  $\pi = 1$  the Witt groups  $W_{-1}(K, \pi, 1)$  and  $W_{-1}(L, \pi, 1)$  are trivial as we remarked earlier in this paper. Our sequences now reduce to those of [3].

*Comment 2.* Note that  $W_{-1}(L, \pi, -) \cong W(L, \pi, -)$  for the reason stated earlier.

Also  $W(D, \pi, \wedge) \cong W_{-1}(D, \pi, -)$  since forms hermitian with respect to  $\wedge$  are equivalent to those skew-hermitian with respect to  $-$  and vice versa. (The correspondence  $\phi \leftrightarrow i\phi$  gives this since  $\hat{x} = i^{-1}\bar{x}i$  for all  $x \in D$ .) A consequence of the above is that the two octagons each display an interesting symmetry

feature. In the “antipodal” position to  $W(F, \pi, J)$  in the octagon we always have  $W_{-1}(F, \pi, J)$ .

*Comment 3.* Our proof is different from that of [3] and it may well be possible that this new method of proof can also be used to generalize the sequences of [3] to the case when  $K$  is a commutative ring and  $L$  is some kind of Galois extension with Galois group cyclic of order two.

#### REFERENCES

- [1] CIBILS, Claude. Groupe de Witt d’une algèbre avec involution. *L’Enseignement Mathématique* 29 (1983), 27-43.
- [2] JACOBSON, N. *Basic Algebra*. W. H. Freeman, San Francisco, 1974.
- [3] LEWIS, D. W. New improved exact sequences of Witt groups. *J. of Algebra* 74 (1982), 206-210.
- [4] ——— A note on hermitian and quadratic forms. *Bull. London Math. Soc.* 11 (1979), 265-267.

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