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space  $M'$  is obtained by gluing, each time along one of the two segments of  $a$  or  $c$ , as many copies of open sectors bounded by the lines  $a$  and  $c$ , (each covering an open annulus [5]) as there are letters in the characteristic word. These projective structures on the 2-torus are characterized by their (cyclic) word and the  $t = 1$  flow map. In suitable homogeneous coordinates the last is expressed as  $f_1 : f_t : (x, y, z) \rightarrow (xe^{\alpha t}, ye^{\beta t}, ze^{\gamma t})$   $\alpha < \beta < \gamma$ ,  $t = 1$ .

*Remark.* Following the curve from its initial point  $P$  to its endpoint  $P'$ , one can say that the sectors of  $P$  and  $P'$  were identified by the identity map: in homogeneous coordinates.

$$(x, y, z) \rightarrow (x, y, z)$$

A more general case (see Goldman [5]) is obtained if we identify by any projectivity commuting with  $f_1$ :

$$g : (x, y, z) \rightarrow (xe^{\lambda}, ye^{\mu}, ze^{\nu})$$

$\lambda, \mu, \nu \in \mathbf{R}$ .

#### AFFINE STRUCTURES IN 2, 3, AND 4 DIMENSIONS

In dimension two only the torus admits an affine structure by Benzecri [1] and for all affine structures the developing map is a covering of its image by Nagano-Yagi [7]. The image is affinely equivalent to either the whole plane, the once punctured plane, the half plane or the quarter plane.

We obtain interesting affine structures in dimensions 3 and 4 using respectively the projective and inversive structures in dimension 2 discussed above.

i) A projective transformation of the real projective plane  $\mathbf{RP}^2 = \mathbf{R}^3 - \{0\}/\mathbf{R}^*$  (where  $\mathbf{R}^* = \mathbf{R} - \{0\}$ ) lifts to an affine transformation of  $V = \mathbf{R}^3 - \{0\}$ , unique but for scalar multiplication. Any such commutes with scalar multiplication by a real number  $\alpha > 1$  (e.g.  $\alpha = 2$ ).

Thus one may build an affine 3-manifold using as a pattern a projective two manifold (open sets in the projective plane lift to open sets (cones) in  $V$  etc.). If we further divide by the action of a compactness is preserved in the construction.

The projective structures on the two torus constructed above yield compact affine 3-manifolds where the developing map is not a covering. In particular, from the example in Figure 7, we can obtain an affine 3-manifold which develops

over the part outside the coordinate axes of  $\{X > 0\} \cup \{Z > 0\} \subset \mathbf{R}^3 = \{(x, y, z)\}$ , but not as a covering. In these examples the 3-manifold  $M$  is a 3-torus.

ii) Similarly, a projective transformation of the complex projective line  $\mathbf{CP}^1 = \mathbf{C}^2 - \{0\}/\mathbf{C}^*$ , that is to say an orientable conformal or inversive transformation of  $S^2 = \mathbf{CP}^1$ , lifts to a complex affine transformation of  $V = \mathbf{C}^2 - \{0\}$ , unique but for scalar multiplication and commuting with scalar multiplication.

We can build a four dimensional affine manifold from an inversive 2-manifold, which is actually a complex affine manifold of  $\mathbf{C}$ -dimension 2, and this construction is the analogue of the above over  $\mathbf{C}$ , thinking of  $S^2$  as  $\mathbf{CP}^1$  and the conformal transformations as the  $\mathbf{C}$ -projective transformation.

Again compactness is achieved if we divide by  $\alpha = 2$ . Thus using the inversive Example 2 we obtain affine 4-manifolds whose developing image has a complicated boundary related to the non-differentiable Jordan curve. Using Example 3, we obtain an affine four-manifold whose developing image in  $R^4$  omits a Cantor set of two planes passing through the origin.

Using Example 4, we can build affine manifolds whose developing map is not a covering of its image (which is all of  $\mathbf{C}^2 - 0$ ). And we repeat, all the above are actually complex affine structures on compact 4-manifolds.

NOTE 1 (see page 16). *Ehresmann* defined the *development map* as follows. Let  $\mathcal{P} \rightarrow M$  be the principle  $\mathcal{A}$ -bundle over  $M$ , whose points are germs  $[x, \kappa]$  of canonical charts  $\{x \in U \subset M, \kappa: U \rightarrow A\}$ . Define a new topology  $\mathcal{F}(\mathcal{P})$  in the set  $\mathcal{P}$  by taking as open set the germs at all points  $x \in U$  of any given chart  $\kappa: U \rightarrow A$ . The natural map  $d: \mathcal{F}(\mathcal{P}) \rightarrow A$  is an immersion. Choose one component of  $\mathcal{F}(\mathcal{P})$  and call it  $M'$ . The restriction  $d: M' \rightarrow A$  is a development map. The restriction of the natural fibre bundle projection  $p: \mathcal{F}(\mathcal{P}) \rightarrow M$  is a covering  $M' \rightarrow M$ .

NOTE 2 (see page 16). *The fibre bundle picture*. For the simple *local* discussion of *one canonical chart*  $U \subset A$ , we can describe a trivial fibre bundle  $E_U = U \times A \rightarrow U$  by assigning to any  $x \in U$  the "heavily osculating" model space  $A_x = A$ . The manifold  $U$  is embedded as the diagonal cross section.  $s(U) = \{(y, y)\} = \text{diag}(U \times U) \subset U \times U \subset U \times A$ . Its points are the points of tangency of fibre and base manifolds. Finally a foliation  $\mathcal{F}$  is defined as the one with horizontal leaves  $U \times \{v\} \subset E_U = U \times A$ , for  $v \in A$ .

For the *global* discussion of an  $\mathcal{A}$ -structure on a manifold  $M$ , we assume  $\mathcal{A}$ -compatible canonical charts that are topological embeddings  $\kappa: U \hookrightarrow A$  for

small open sets  $U \subset M$ . A point of the fibre bundle space  $E$  over  $M$  is by definition a triple

$$\{x, \kappa, v\},$$

where  $x \in U \subset M$ ,  $\kappa: U \rightarrow A$  is a canonical chart and  $v \in A$ , *modulo equivalence* by the action of  $\mathcal{A}$  given by  $g: \{x, \kappa, v\} \simeq \{x, \kappa', v'\}$  where  $\kappa' = g \circ \kappa$ ,  $v' = gv$ ,  $g \in \mathcal{A}$ . In  $E$ ,  $M$  is embedded as the "diagonal cross section"  $s(M)$ , whose points are represented by triples  $\{x, \kappa, \kappa(x)\}$ . The foliation  $\mathcal{F}$  has the local "horizontal" leaves represented by triples  $\{U, \kappa, v\}$ . For contractible closed curves starting and ending at  $x_0 \in M$  in the base space  $M$ , the holonomy of the foliation is of course the identity map of the fibre  $A_{x_0}$ . As a consequence for closed curves in general, starting and ending at  $x_0$  the holonomy gives the representation of  $\pi_1 M$  into the group  $\mathcal{A}$  acting on  $A_{x_0}$ . "Parallel displacement" of the points of  $s(M)$  along the lifting in  $\mathcal{F}$ -leaves of curves in the base space ending at  $x_0$ , determines the development map  $M' \rightarrow A_{x_0}$ .

NOTE 3 (see page 16). *Flat Cartan connections.* Manifolds with canonical  $(\mathcal{A}, A)$ -charts are the flat cases (without torsion and without curvature) of manifolds  $M$  with a general  $(\mathcal{A}, A)$ -connection. They are defined in [4] as follows

- (1) A fibre bundle  $A \rightarrow E \rightarrow M$  with fibre  $A$  over  $M$
- (2) A fixed cross section  $s(M)$
- (3) An  $n$ -plane field  $\xi$  in  $E$  transversal to the fibres and transversal to the fixed cross sections, such that
- (4) The holonomy obtained by lifting a closed curve starting and ending at  $x_0 \in M$ , into all curves tangent to  $\xi$ , belongs to  $\mathcal{A}$  acting on  $A_{x_0}$ . It is in general different for homotopic curves. It is flat if contractible closed curves have trivial holonomy (= identity).

The development of a curve ending in  $x_0$  in  $M$ , is obtained by dragging along  $\xi$  the corresponding points of  $s(M)$  until they arrive in the fibre  $A_{x_0}$ . In the flat case homotopic curves with common initial and end points give the same image of the initial point in the end fibre and the development map is achieved.