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space M' is obtained by gluing, each time along one of the two segments of a or c , as many copies of open sectors bounded by the lines a and c , (each covering an open annulus [5]) as there are letters in the characteristic word. These projective structures on the 2-torus are characterized by their (cyclic) word and the $t = 1$ flow map. In suitable homogeneous coordinates the last is expressed as $f_1 : f_t : (x, y, z) \rightarrow (xe^{\alpha t}, ye^{\beta t}, ze^{\gamma t})$ $\alpha < \beta < \gamma$, $t = 1$.

Remark. Following the curve from its initial point P to its endpoint P' , one can say that the sectors of P and P' were identified by the identity map: in homogeneous coordinates.

$$(x, y, z) \rightarrow (x, y, z)$$

A more general case (see Goldman [5]) is obtained if we identify by any projectivity commuting with f_1 :

$$g : (x, y, z) \rightarrow (xe^{\lambda}, ye^{\mu}, ze^{\nu})$$

$\lambda, \mu, \nu \in \mathbf{R}$.

AFFINE STRUCTURES IN 2, 3, AND 4 DIMENSIONS

In dimension two only the torus admits an affine structure by Benzecri [1] and for all affine structures the developing map is a covering of its image by Nagano-Yagi [7]. The image is affinely equivalent to either the whole plane, the once punctured plane, the half plane or the quarter plane.

We obtain interesting affine structures in dimensions 3 and 4 using respectively the projective and inversive structures in dimension 2 discussed above.

i) A projective transformation of the real projective plane $\mathbf{RP}^2 = \mathbf{R}^3 - \{0\}/\mathbf{R}^*$ (where $\mathbf{R}^* = \mathbf{R} - \{0\}$) lifts to an affine transformation of $V = \mathbf{R}^3 - \{0\}$, unique but for scalar multiplication. Any such commutes with scalar multiplication by a real number $\alpha > 1$ (e.g. $\alpha = 2$).

Thus one may build an affine 3-manifold using as a pattern a projective two manifold (open sets in the projective plane lift to open sets (cones) in V etc.). If we further divide by the action of a compactness is preserved in the construction.

The projective structures on the two torus constructed above yield compact affine 3-manifolds where the developing map is not a covering. In particular, from the example in Figure 7, we can obtain an affine 3-manifold which develops

over the part outside the coordinate axes of $\{X > 0\} \cup \{Z > 0\} \subset \mathbf{R}^3 = \{(x, y, z)\}$, but not as a covering. In these examples the 3-manifold M is a 3-torus.

ii) Similarly, a projective transformation of the complex projective line $\mathbf{CP}^1 = \mathbf{C}^2 - \{0\}/\mathbf{C}^*$, that is to say an orientable conformal or inversive transformation of $S^2 = \mathbf{CP}^1$, lifts to a complex affine transformation of $V = \mathbf{C}^2 - \{0\}$, unique but for scalar multiplication and commuting with scalar multiplication.

We can build a four dimensional affine manifold from an inversive 2-manifold, which is actually a complex affine manifold of \mathbf{C} -dimension 2, and this construction is the analogue of the above over \mathbf{C} , thinking of S^2 as \mathbf{CP}^1 and the conformal transformations as the \mathbf{C} -projective transformation.

Again compactness is achieved if we divide by $\alpha = 2$. Thus using the inversive Example 2 we obtain affine 4-manifolds whose developing image has a complicated boundary related to the non-differentiable Jordan curve. Using Example 3, we obtain an affine four-manifold whose developing image in R^4 omits a Cantor set of two planes passing through the origin.

Using Example 4, we can build affine manifolds whose developing map is not a covering of its image (which is all of $\mathbf{C}^2 - 0$). And we repeat, all the above are actually complex affine structures on compact 4-manifolds.

NOTE 1 (see page 16). *Ehresmann* defined the *development map* as follows. Let $\mathcal{P} \rightarrow M$ be the principle \mathcal{A} -bundle over M , whose points are germs $[x, \kappa]$ of canonical charts $\{x \in U \subset M, \kappa: U \rightarrow A\}$. Define a new topology $\mathcal{F}(\mathcal{P})$ in the set \mathcal{P} by taking as open set the germs at all points $x \in U$ of any given chart $\kappa: U \rightarrow A$. The natural map $d: \mathcal{F}(\mathcal{P}) \rightarrow A$ is an immersion. Choose one component of $\mathcal{F}(\mathcal{P})$ and call it M' . The restriction $d: M' \rightarrow A$ is a development map. The restriction of the natural fibre bundle projection $p: \mathcal{F}(\mathcal{P}) \rightarrow M$ is a covering $M' \rightarrow M$.

NOTE 2 (see page 16). *The fibre bundle picture*. For the simple *local* discussion of *one canonical chart* $U \subset A$, we can describe a trivial fibre bundle $E_U = U \times A \rightarrow U$ by assigning to any $x \in U$ the "heavily osculating" model space $A_x = A$. The manifold U is embedded as the diagonal cross section. $s(U) = \{(y, y)\} = \text{diag}(U \times U) \subset U \times U \subset U \times A$. Its points are the points of tangency of fibre and base manifolds. Finally a foliation \mathcal{F} is defined as the one with horizontal leaves $U \times \{v\} \subset E_U = U \times A$, for $v \in A$.

For the *global* discussion of an \mathcal{A} -structure on a manifold M , we assume \mathcal{A} -compatible canonical charts that are topological embeddings $\kappa: U \hookrightarrow A$ for

small open sets $U \subset M$. A point of the fibre bundle space E over M is by definition a triple

$$\{x, \kappa, v\},$$

where $x \in U \subset M$, $\kappa: U \rightarrow A$ is a canonical chart and $v \in A$, *modulo equivalence* by the action of \mathcal{A} given by $g: \{x, \kappa, v\} \ni \{x, \kappa', v'\}$ where $\kappa' = g \circ \kappa$, $v' = gv$, $g \in \mathcal{A}$. In E , M is embedded as the "diagonal cross section" $s(M)$, whose points are represented by triples $\{x, \kappa, \kappa(x)\}$. The foliation \mathcal{F} has the local "horizontal" leaves represented by triples $\{U, \kappa, v\}$. For contractible closed curves starting and ending at $x_0 \in M$ in the base space M , the holonomy of the foliation is of course the identity map of the fibre A_{x_0} . As a consequence for closed curves in general, starting and ending at x_0 the holonomy gives the representation of $\pi_1 M$ into the group \mathcal{A} acting on A_{x_0} . "Parallel displacement" of the points of $s(M)$ along the lifting in \mathcal{F} -leaves of curves in the base space ending at x_0 , determines the development map $M' \rightarrow A_{x_0}$.

NOTE 3 (see page 16). *Flat Cartan connections.* Manifolds with canonical (\mathcal{A}, A) -charts are the flat cases (without torsion and without curvature) of manifolds M with a general (\mathcal{A}, A) -connection. They are defined in [4] as follows

- (1) A fibre bundle $A \rightarrow E \rightarrow M$ with fibre A over M
- (2) A fixed cross section $s(M)$
- (3) An n -plane field ξ in E transversal to the fibres and transversal to the fixed cross sections, such that
- (4) The holonomy obtained by lifting a closed curve starting and ending at $x_0 \in M$, into all curves tangent to ξ , belongs to \mathcal{A} acting on A_{x_0} . It is in general different for homotopic curves. It is flat if contractible closed curves have trivial holonomy (= identity).

The development of a curve ending in x_0 in M , is obtained by dragging along ξ the corresponding points of $s(M)$ until they arrive in the fibre A_{x_0} . In the flat case homotopic curves with common initial and end points give the same image of the initial point in the end fibre and the development map is achieved.