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EXAMPLES

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space M' is obtained by gluing, each time along one of the two segments of a or c, as many copies of open sectors bounded by the lines a and c, (each covering an open annulus [5]) as there are letters in the characteristic word. These projective structures on the 2-torus are characterized by their (cyclic) word and the t=1 flow map. In suitable homogeneous coordinates the last is expressed as $f_1: f_t: (x, y, z) \to (xe^{\alpha t}, ye^{\beta t}, ze^{\gamma t}) \ \alpha < \beta < \gamma, \quad t=1$.

Remark. Following the curve from its initial point P to its endpoint P', one can say that the sectors of P and P' were identified by the identity map: in homogeneous coordinates.

$$(x, y, z) \rightarrow (x, y, z)$$

A more general case (see Goldman [5]) is obtained if we identify by any projectivity commuting with f_1 :

$$g:(x, y, z) \rightarrow (xe^{\lambda}, ye^{\mu}, ze^{\nu})$$

 $\lambda, \mu, \nu \in \mathbf{R}$.

Affine structures in 2, 3, and 4 dimensions

In dimension two only the torus admits an affine structure by Benzecri [1] and for all affine structures the developing map is a covering of its image by Nagano-Yagi [7]. The image is affinely equivalent to either the whole plane, the once punctured plane, the half plane or the quarter plane.

We obtain interesting affine structures in dimensions 3 and 4 using respectively the projective and inversive structures in dimension 2 discussed above.

- i) A projective transformation of the real projective plane $\mathbf{RP}^2 = \mathbf{R}^3$
- $-\{0\}/\mathbb{R}^*$ (where $\mathbb{R}^* = \mathbb{R} \{0\}$) lifts to an affine transformation of $V = \mathbb{R}^3$
- $-\{0\}$, unique but for scalar multiplication. Any such commutes with scalar multiplication by a real number $\alpha > 1$ (e.g. $\alpha = 2$).

Thus one may build an affine 3-manifold using as a pattern a projective two manifold (open sets in the projective plane lift to open sets (cones) in V etc.). If we further divide by the action of a compactness is preserved in the construction.

The projective structures on the two torus constructed above yield compact affine 3-manifolds where the developing map is not a covering. In particular, from the example in Figure 7, we can obtain an affine 3-manifold which develops

over the part outside the coordinate axes of $\{X > 0\} \cup \{Z > 0\} \subset \mathbb{R}^3$ = $\{(x, y, z)\}$, but not as a covering. In these examples the 3-manifold M is a 3-torus.

ii) Similarly, a projective transformation of the complex projective line $\mathbb{CP}^1 = \mathbb{C}^2 - \{0\}/\mathbb{C}^*$, that is to say an orientable conformal or inversive transformation of $S^2 = \mathbb{CP}^1$, lifts to a complex affine transformation of $V = \mathbb{C}^2 - \{0\}$, unique but for scalar multiplication and commuting with scalar multiplication.

We can build a four dimensional affine manifold from an inversive 2-manifold, which is actually a complex affine manifold of C-dimension 2, and this construction is the analogue of the above over C, thinking of S^2 as \mathbb{CP}^1 and the conformal transformations as the C-projective transformation.

Again compactness is achieved if we divide by $\alpha = 2$. Thus using the inversive Example 2 we obtain affine 4-manifolds whose developing image has a complicated boundary related to the non-differentiable Jordan curve. Using Example 3, we obtain an affine four-manifold whose developing image in R^4 omits a Cantor set of two planes passing through the origin.

Using Example 4, we can build affine manifolds whose developing map is not a covering of its image (which is all of $\mathbb{C}^2 - 0$). And we repeat, all the above are actually complex affine structures on compact 4-manifolds.

NOTE 1 (see page 16). Ehresmann defined the development map as follows. Let $\mathscr{P} \to M$ be the principle \mathscr{A} -bundle over M, whose points are germs $[x, \kappa]$ of canonical charts $\{x \in U \subset M, \kappa \colon U \to A\}$. Define a new topology $\mathscr{F}(\mathscr{P})$ in the set \mathscr{P} by taking as open set the germs at all points $x \in U$ of any given chart $\kappa \colon U \to A$. The natural map $d \colon \mathscr{F}(\mathscr{P}) \to A$ is an immersion. Choose one component of $\mathscr{F}(\mathscr{P})$ and call it M'. The restriction $d \colon M' \to A$ is a development map. The restriction of the natural fibre bundle projection $p \colon \mathscr{F}(\mathscr{P}) \to M$ is a covering $M' \to M$.

NOTE 2 (see page 16). The fibre bundle picture. For the simple local discussion of one canonical chart $U \subset A$, we can describe a trivial fibre bundle $E_U = U \times A \to U$ by assigning to any $x \in U$ the "heavily osculating" model space $A_x = A$. The manifold U is embedded as the diagonal cross section. $s(U) = \{(y, y)\} = \text{diag}(U \times U) \subset U \times U \subset U \times A$. Its points are the points of tangency of fibre and base manifolds. Finally a foliation \mathscr{F} is defined as the one with horizontal leaves $U \times \{v\} \subset E_U = U \times A$, for $v \in A$.

For the *global* discussion of an \mathscr{A} -structure on a manifold M, we assume \mathscr{A} -compatible canonical charts that are topological embeddings $\kappa: U \hookrightarrow A$ for

small open sets $U \subset M$. A point of the fibre bundle space E over M is by definition a triple

$$\{x, \kappa, v\}$$
,

where $x \in U \subset M$, $\kappa: U \to A$ is a canonical chart and $v \in A$, modulo equivalence by the action of \mathscr{A} given by $g: \{x, \kappa, v\} \cong \{x, \kappa', v'\}$ where $\kappa' = g \circ \kappa$, v' = gv, $g \in \mathscr{A}$. In E, M is embedded as the "diagonal cross section" s(M), whose points are represented by triples $\{x, \kappa, \kappa(x)\}$. The foliation \mathscr{F} has the local "horizontal" leaves represented by triples $\{U, \kappa, v\}$. For contractible closed curves starting and ending at $x_0 \in M$ in the base space M, the holonomy of the foliation is of course the identity map of the fibre A_{κ_0} . As a consequence for closed curves in general, starting and ending at κ_0 the holonomy gives the representation of $\kappa_1 M$ into the group \mathscr{A} acting on κ_0 . "Parallel displacement" of the points of κ_0 along the lifting in \mathscr{F} -leaves of curves in the base space ending at κ_0 , determines the development map $\kappa_0 \to \kappa_0$.

NOTE 3 (see page 16). Flat Cartan connections. Manifolds with canonical (\mathcal{A}, A) -charts are the flat cases (without torsion and without curvature) of manifolds M with a general (\mathcal{A}, A) -connection. They are defined in [4] as follows

- (1) A fibre bundle $A \to E \to M$ with fibre A over M
- (2) A fixed cross section s(M)
- (3) An *n*-plane field ξ in *E* transversal to the fibres and transversal to the fixed cross sections, such that
- (4) The holonomy obtained by lifting a closed curve starting and ending at $x_0 \in M$, into all curves tangent to ξ , belongs to \mathscr{A} acting on A_{x_0} . It is in general different for homotopic curves. It is flat if contractible closed curves have trivial holonomy (= identity).

The development of a curve ending in x_0 in M, is obtained by dragging along ξ the corresponding points of s(M) until they arrive in the fibre A_{x_0} . In the flat case homotopic curves with common initial and end points give the same image of the initial point in the end fibre and the development map is achieved.