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Artikel: MANIFOLDS WITH CANONICAL COORDINATE CHARTS : SOME EXAMPLES

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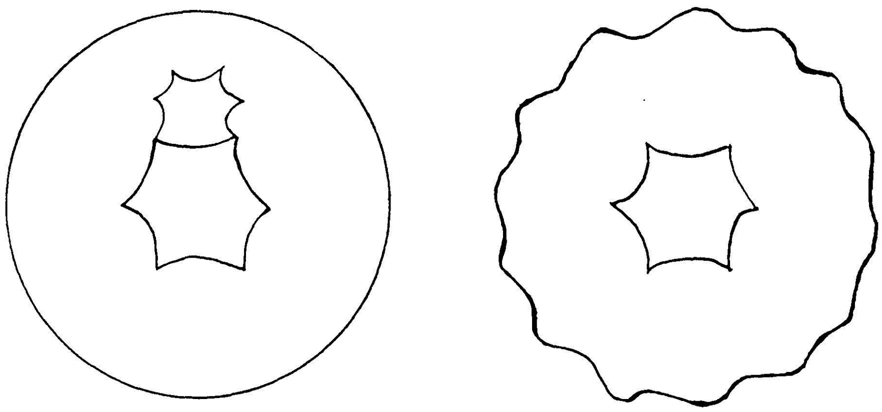
Inversive 2-manifolds

Inversive structures on orientable two manifolds of genus >1 form a rich theory properly containing for example the classical subject of Fuchsian and Kleinian surface groups.

If $Sl(2, \mathbf{C})/\pm 1 = Gl(2, \mathbf{C})/Gl(1, \mathbf{C})$ is the group of fractional linear transformations of $\mathbf{CP}^1$, that is the group of orientable inversive (conformal) transformations of S^2, and Γ is a discrete subgroup acting freely and discontinuously on a connected open set $\Omega \subset S^2$, then Ω/Γ is a 2-manifold M with inversive structure. M' is just Ω and the developing map is an embedding.

Example 1. If Γ is a Fuchsian group, that is, Ω is an open (round) disk in $\mathbf{C} \subset S^2$, then the inversive structure is actually a hyperbolic structure—corresponding to a metric of constant negative curvature. The structure is inversive and projective at the same time.

Example 2. If Γ as in Example 1 is deformed slightly (a so-called quasi-Fuchsian group; see [9]) then Ω remains an open disk whose boundary can be a rather remarkable non rectifiable Jordan curve. This curve has no tangent at a dense set.



Example 1 Example 2

FIGURE 2

Example 3. Let Γ be generated by two general hyperbolic elements of sufficient strength so that the union of the fundamental domains of each covers the entire sphere. Then Ω is S^2 minus a Cantor set and Ω/F is a compact conformal 2 manifold whose developing image is Ω. (Shottky group)
In Figure 3, r_1, r_2 and r_3 are inversions (reflections) in three circles and Γ consists of all products of an even number of these inversions. Γ is generated by r_1r_2 and r_1r_3. A fundamental domain is $D \cup r_1D$, $D = D_1 \cup D_2$. The Cantor set appears clearly on the line of symmetry m.

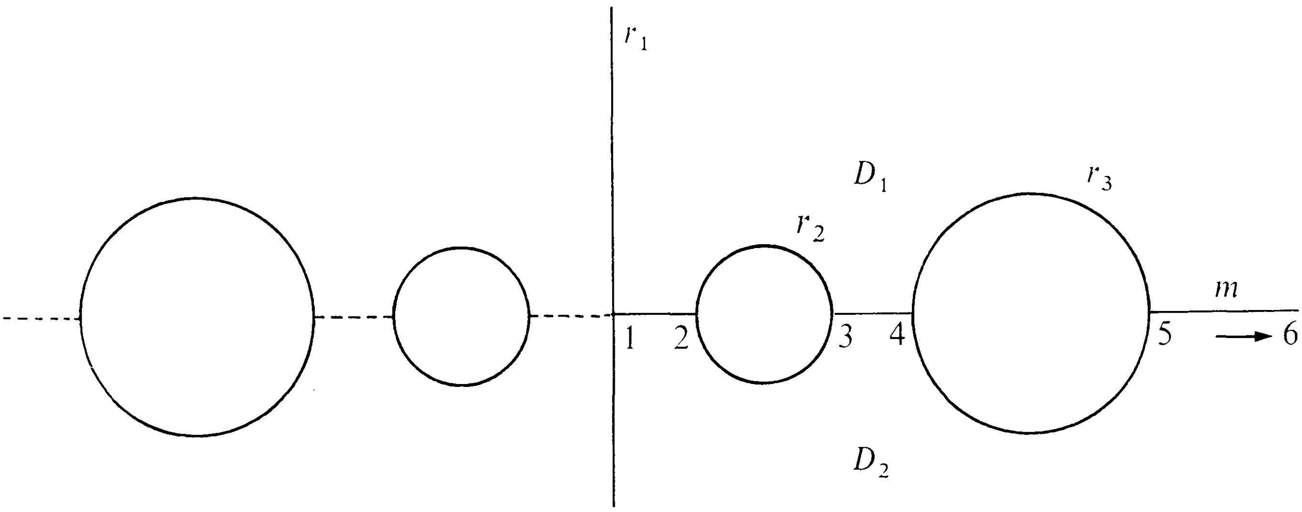


FIGURE 3

Example 4. A class of examples not always arising from Kleinian groups as above can be achieved as follows. Let γ be the boundary of an immersed disk in S^2. Approximate γ by a closed immersed curve again bounding an immersed disk constituted of $2g + 2$ (for some integer $g > 0$) successive arcs of circles meeting at right acute angles (Fig. 4). The new disk with scalloped edges has a conformal structure from the immersion and four of these may be assembled to obtain an inversive 2-manifold of genus g. This *topological* assemblage is suggested in Figure 5.

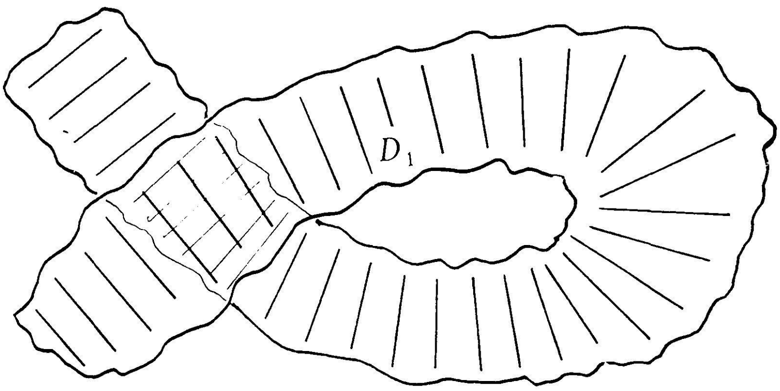


FIGURE 4

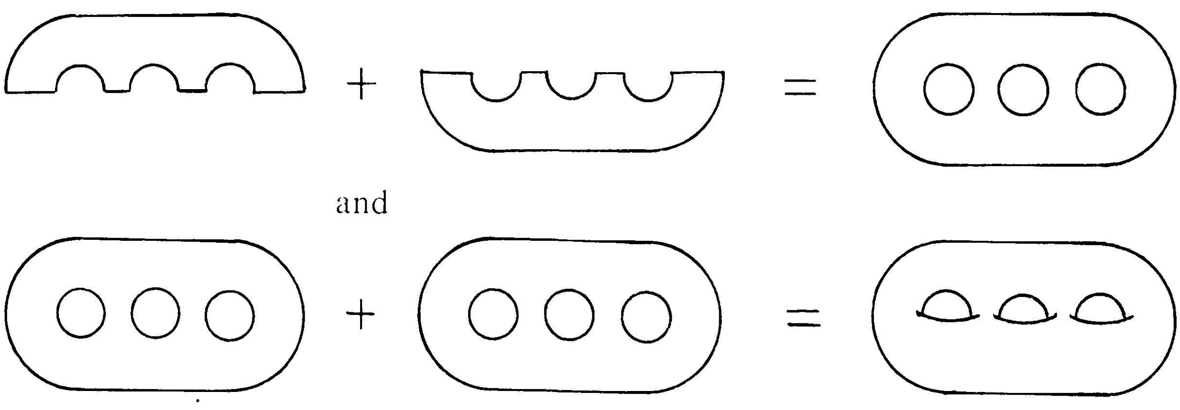


FIGURE 5

Note this construction uses inversion in circles, and four angles at a vertex add up to achieve the non singular conformal structure. Also note the original immersed disk may be chosen (for g big enough) to cover S^2 completely (in a very complicated way) and then the developing map $M' \to S^2$ cannot be a covering. In Figure 6 an example with immersed disk D with 6 vertices ($g = 2$) is suggested, where the developing map covers clearly S^2 completely.

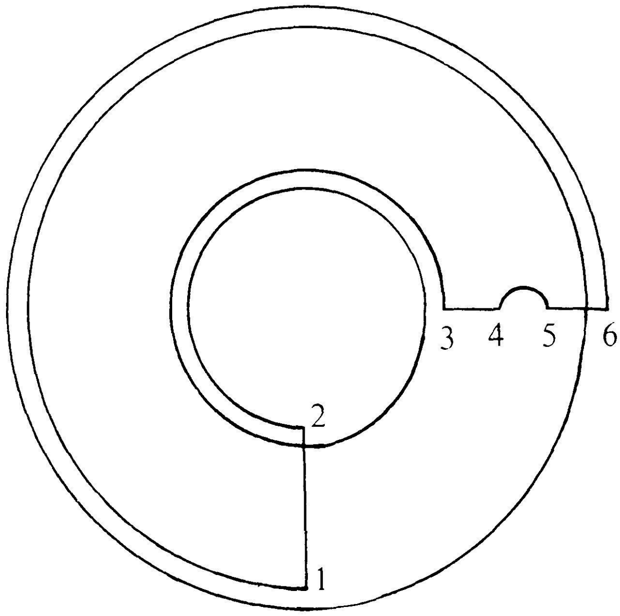


FIGURE 6

We note conversely that if the developing map $M' \to S^2$ is not onto (see Fig. 3, where D_1 is the initial disk, for an example) then the developing map is rather remarkably a covering of its image (Gunning [6]). The idea of the proof is the following—if the image omits at least three points, (exactly one or two points is easy) M' has a Poincaré metric of constant negative curvature preserved by the holonomy group of Moebius transformation acting on the image. Then the developing map becomes an isometric immersion of a complete manifold and thus a covering map.

Example 5. There are interesting projective structures on the torus constructed as follows. Start with a *generic* linear flow on the projective plane (with a source, a sink, and a saddle in point B in Fig. 7a) and choose an immersed curve transverse to the flow lines (Fig. 7b). Note that such curves may be based on a word in 2 symbols for example *ccaaaa* in Figure 7, and *ccaaacacaa* in Figure 8, where the closed curve on $\mathbf{RP}^2$ is drawn on the open band that universally covers the Moebius band, projective plane minus point B.

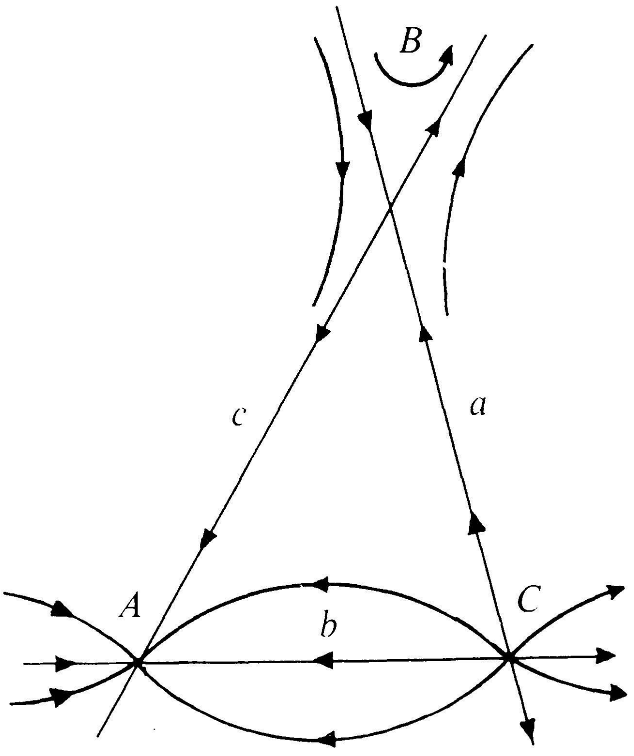


FIGURE 7*a*

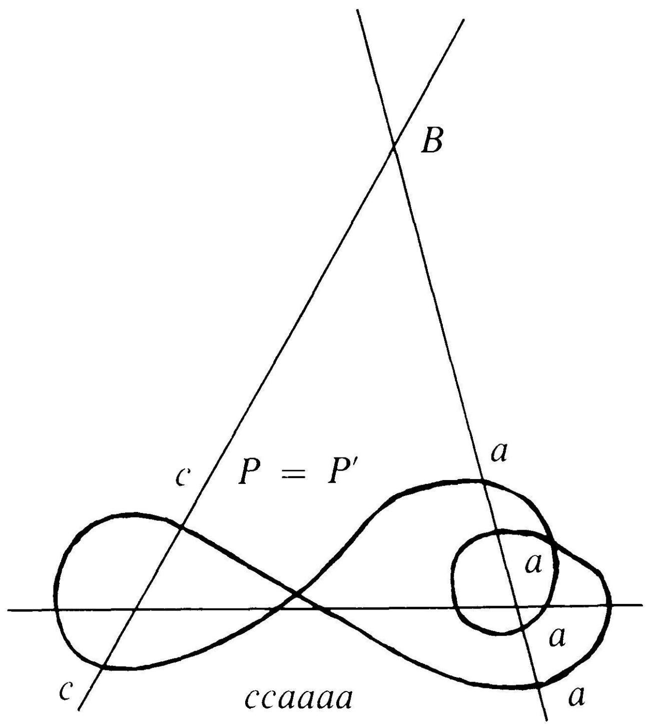


FIGURE 7*b*

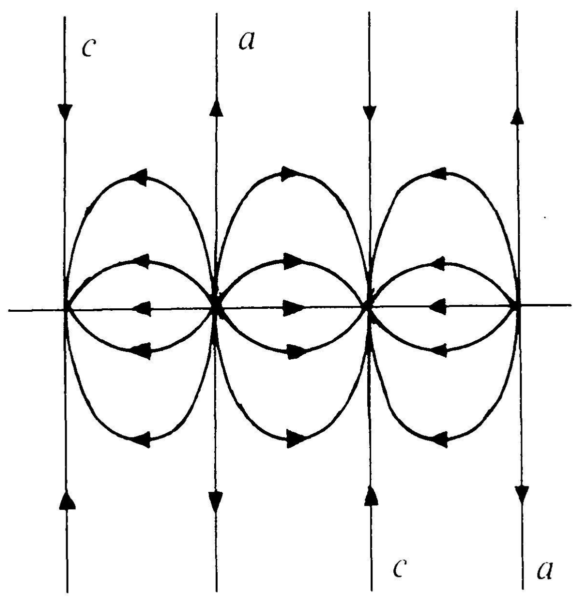


FIGURE 8*a*

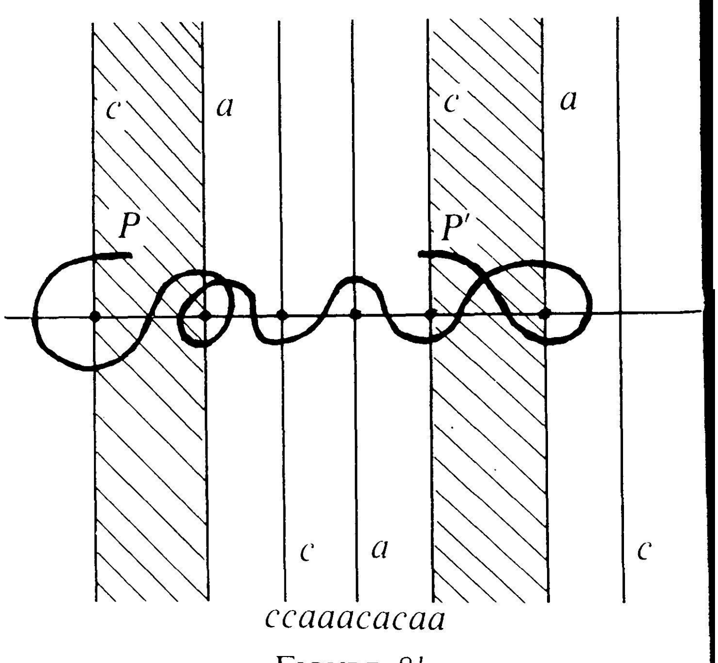


FIGURE 8*b*

Flowing the curve along for time t sweeps out a thickening of the immersed curve, an immersed annulus. We may identify the two boundary components of the annulus by the time t map, a locally projective isomorphism.

The identification space is a projective structure on the torus M whose developing map is the map: $M' = S^1 \times R \to \mathbf{RP}^2$, obtained by spreading the immersed curve around by the flow for all time $t \in \mathbf{R}$.

The developing map is not a covering and the image is the projective plane minus three points for any word different from *aa* or *cc*. Note that the covering space M' is obtained by gluing, each time along one of the two segments of a or c, as many copies of open sectors bounded by the lines a and c, (each covering an open annulus [5]) as there are letters in the characteristic word. These projective structures on the 2-torus are characterized by their (cyclic) word and the $t = 1$ flow map. In suitable homogeneous coordinates the last is expressed as $f_1 : f_t : (x, y, z) \to (xe^{\alpha t}, ye^{\beta t}, ze^{\gamma t})$ $\alpha < \beta < \gamma$, $t = 1$.

Remark. Following the curve from its initial point P to its endpoint P', one can say that the sectors of P and P' were identified by the identity map: in homogeneous coordinates.

$$(x, y, z) \to (x, y, z)$$

A more general case (see Goldman [5]) is obtained if we identify by any projectivity commuting with f_1:

$$g : (x, y, z) \to (xe^{\lambda}, ye^{\mu}, ze^{\nu})$$

$\lambda, \mu, \nu \in \mathbf{R}$.

Affine structures in 2, 3, and 4 dimensions

In dimension two only the torus admits an affine structure by Benzecri [1] and for all affine structures the developing map is a covering of its image by Nagano-Yagi [7]. The image is affinely equivalent to either the whole plane, the once punctured plane, the half plane or the quarter plane.

We obtain interesting affine structures in dimensions 3 and 4 using respectively the projective and inversive structures in dimension 2 discussed above.

i) A projective transformation of the real projective plane $\mathbf{RP}^2 = \mathbf{R}^3 - \{0\}/\mathbf{R}^*$ (where $\mathbf{R}^* = \mathbf{R} - \{0\}$) lifts to an affine transformation of $V = \mathbf{R}^3 - \{0\}$, unique but for scalar multiplication. Any such commutes with scalar multiplication by a real number $\alpha > 1$ (e.g. $\alpha = 2$).

Thus one may build an affine 3-manifold using as a pattern a projective two manifold (open sets in the projective plane lift to open sets (cones) in V etc.). If we further divide by the action of a compactness is preserved in the construction.

The projective structures on the two torus constructed above yield compact affine 3-manifolds where the developing map is not a covering. In particular, from the example in Figure 7, we can obtain an affine 3-manifold which develops