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Artikel: MANIFOLDS WITH CANONICAL COORDINATE CHARTS : SOME EXAMPLES

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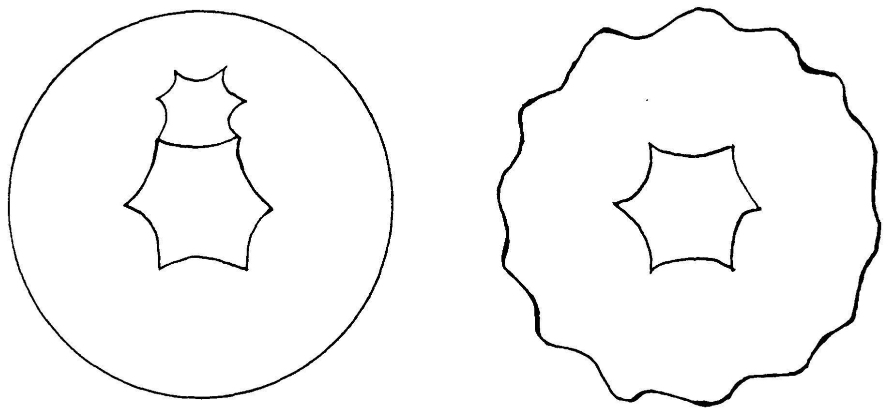
INVERSIVE 2-MANIFOLDS

Inversive structures on orientable two manifolds of genus >1 form a rich theory properly containing for example the classical subject of Fuchsian and Kleinian surface groups.

If $Sl(2, \mathbf{C})/\pm 1 = Gl(2, \mathbf{C})/Gl(1, \mathbf{C})$ is the group of fractional linear transformations of $\mathbf{CP}^1$, that is the group of orientable inversive (conformal) transformations of S^2, and Γ is a discrete subgroup acting freely and discontinuously on a connected open set $\Omega \subset S^2$, then Ω/Γ is a 2-manifold M with inversive structure. M' is just Ω and the developing map is an embedding.

Example 1. If Γ is a Fuchsian group, that is, Ω is an open (round) disk in $\mathbf{C} \subset S^2$, then the inversive structure is actually a hyperbolic structure—corresponding to a metric of constant negative curvature. The structure is inversive and projective at the same time.

Example 2. If Γ as in Example 1 is deformed slightly (a so-called quasi-Fuchsian group; see [9]) then Ω remains an open disk whose boundary can be a rather remarkable non rectifiable Jordan curve. This curve has no tangent at a dense set.



Example 1 Example 2

FIGURE 2

Example 3. Let Γ be generated by two general hyperbolic elements of sufficient strength so that the union of the fundamental domains of each covers the entire sphere. Then Ω is S^2 minus a Cantor set and Ω/F is a compact conformal 2 manifold whose developing image is Ω. (Shottky group)
In Figure 3, r_1, r_2 and r_3 are inversions (reflections) in three circles and Γ consists of all products of an even number of these inversions. Γ is generated by $r_1 r_2$ and $r_1 r_3$. A fundamental domain is $D \cup r_1 D$, $D = D_1 \cup D_2$. The Cantor set appears clearly on the line of symmetry m.

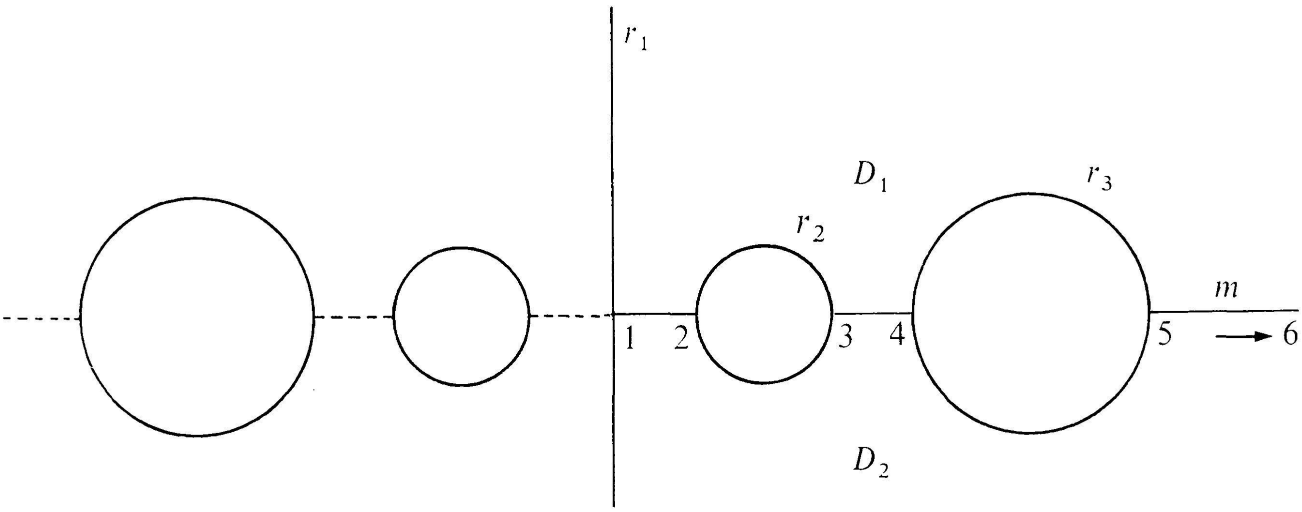


FIGURE 3

Example 4. A class of examples not always arising from Kleinian groups as above can be achieved as follows. Let γ be the boundary of an immersed disk in S^2. Approximate γ by a closed immersed curve again bounding an immersed disk constituted of $2g + 2$ (for some integer $g > 0$) successive arcs of circles meeting at right acute angles (Fig. 4). The new disk with scalloped edges has a conformal structure from the immersion and four of these may be assembled to obtain an inversive 2-manifold of genus g. This *topological* assemblage is suggested in Figure 5.

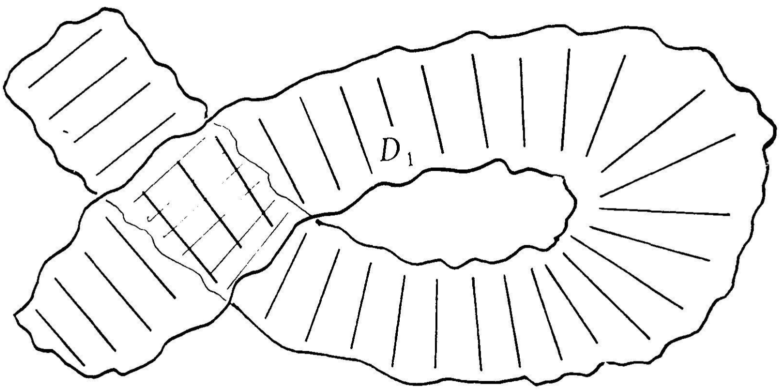


FIGURE 4

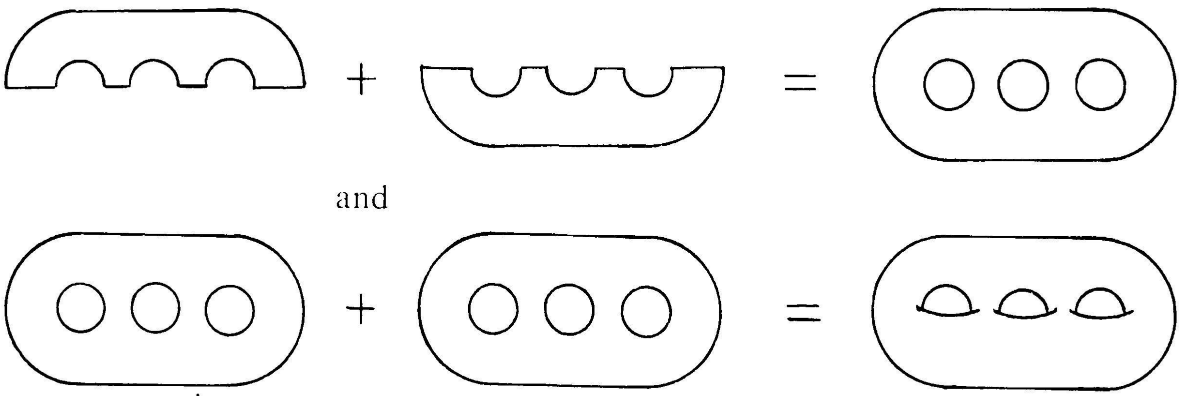


FIGURE 5

Note this construction uses inversion in circles, and four angles at a vertex add up to achieve the non singular conformal structure. Also note the original immersed disk may be chosen (for g big enough) to cover S^2 completely (in a very complicated way) and then the developing map $M' \to S^2$ cannot be a covering. In Figure 6 an example with immersed disk D with 6 vertices ($g = 2$) is suggested, where the developing map covers clearly S^2 completely.

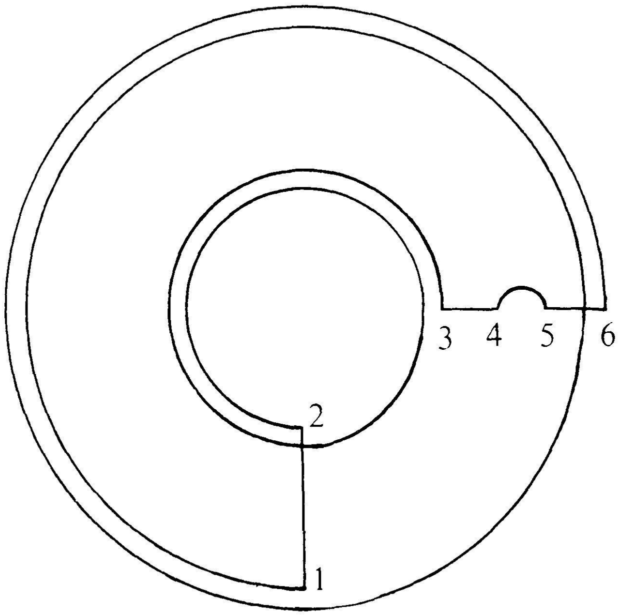


FIGURE 6

We note conversely that if the developing map $M' \to S^2$ is not onto (see Fig. 3, where D_1 is the initial disk, for an example) then the developing map is rather remarkably a covering of its image (Gunning [6]). The idea of the proof is the following—if the image omits at least three points, (exactly one or two points is easy) M' has a Poincaré metric of constant negative curvature preserved by the holonomy group of Moebius transformation acting on the image. Then the developing map becomes an isometric immersion of a complete manifold and thus a covering map.

Example 5. There are interesting projective structures on the torus constructed as follows. Start with a *generic* linear flow on the projective plane (with a source, a sink, and a saddle in point B in Fig. 7a) and choose an immersed curve transverse to the flow lines (Fig. 7b). Note that such curves may be based on a word in 2 symbols for example *ccaaaa* in Figure 7, and *ccaaacacaa* in Figure 8, where the closed curve on $\mathbf{RP}^2$ is drawn on the open band that universally covers the Moebius band, projective plane minus point B.

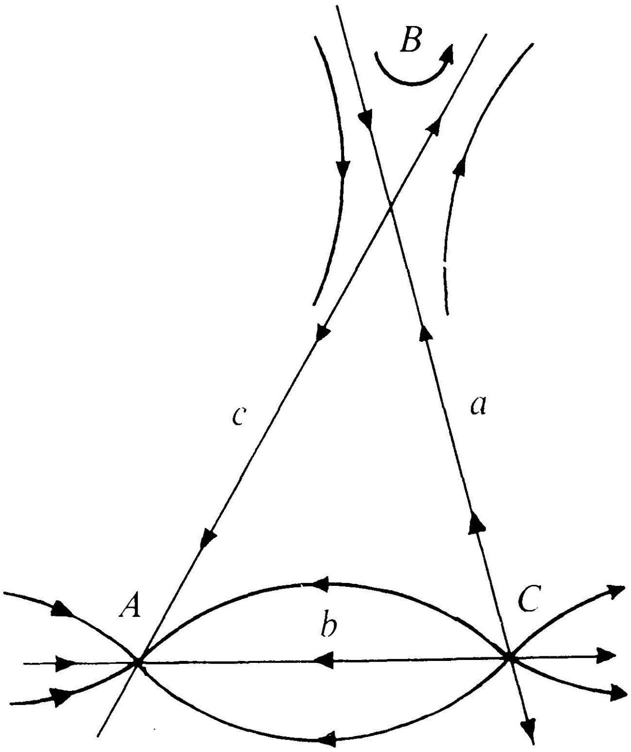


FIGURE 7a

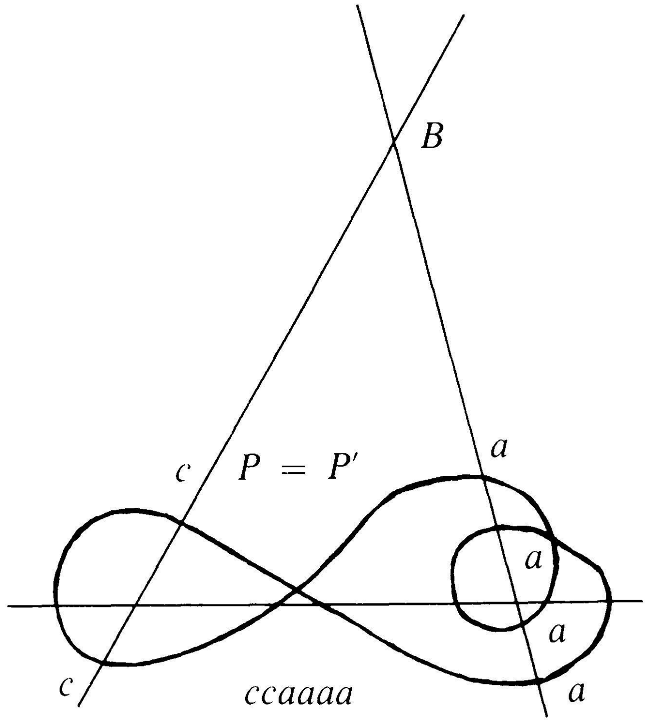


FIGURE 7b

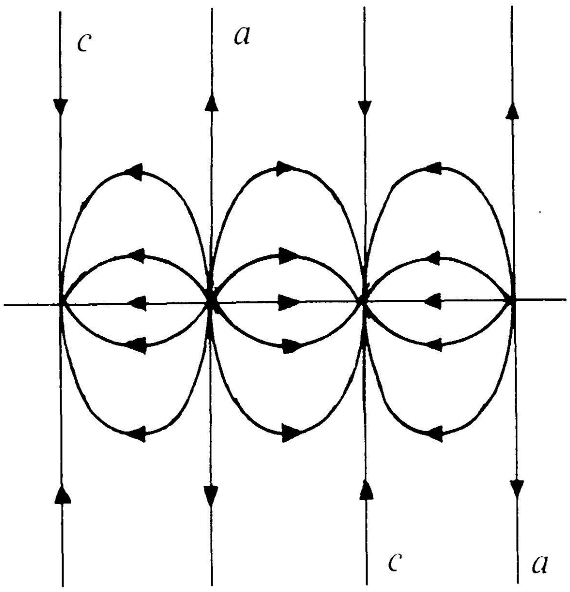


FIGURE 8a

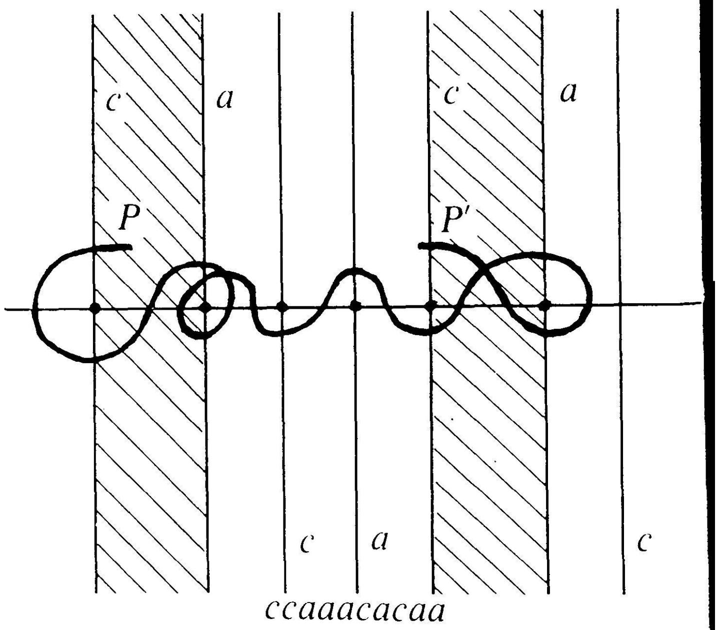


FIGURE 8b

Flowing the curve along for time t sweeps out a thickening of the immersed curve, an immersed annulus. We may identify the two boundary components of the annulus by the time t map, a locally projective isomorphism.

The identification space is a projective structure on the torus M whose developing map is the map: $M' = S^1 \times R \to \mathbf{RP}^2$, obtained by spreading the immersed curve around by the flow for all time $t \in \mathbf{R}$.

The developing map is not a covering and the image is the projective plane minus three points for any word different from aa or cc. Note that the covering space M' is obtained by gluing, each time along one of the two segments of a or c, as many copies of open sectors bounded by the lines a and c, (each covering an open annulus [5]) as there are letters in the characteristic word. These projective structures on the 2-torus are characterized by their (cyclic) word and the $t = 1$ flow map. In suitable homogeneous coordinates the last is expressed as $f_1 : f_t : (x, y, z) \to (xe^{\alpha t}, ye^{\beta t}, ze^{\gamma t})\ \alpha < \beta < \gamma, \quad t = 1.$

Remark. Following the curve from its initial point P to its endpoint P', one can say that the sectors of P and P' were identified by the identity map: in homogeneous coordinates.

$$(x, y, z) \to (x, y, z)$$

A more general case (see Goldman [5]) is obtained if we identify by any projectivity commuting with f_1:

$$g : (x, y, z) \to (xe^{\lambda}, ye^{\mu}, ze^{\nu})$$

$\lambda, \mu, \nu \in \mathbf{R}$.

AFFINE STRUCTURES IN 2, 3, AND 4 DIMENSIONS

In dimension two only the torus admits an affine structure by Benzecri [1] and for all affine structures the developing map is a covering of its image by Nagano-Yagi [7]. The image is affinely equivalent to either the whole plane, the once punctured plane, the half plane or the quarter plane.

We obtain interesting affine structures in dimensions 3 and 4 using respectively the projective and inversive structures in dimension 2 discussed above.

i) A projective transformation of the real projective plane $\mathbf{RP}^2 = \mathbf{R}^3 - \{0\}/\mathbf{R}^*$ (where $\mathbf{R}^* = \mathbf{R} - \{0\}$) lifts to an affine transformation of $V = \mathbf{R}^3 - \{0\}$, unique but for scalar multiplication. Any such commutes with scalar multiplication by a real number $\alpha > 1$ (e.g. $\alpha = 2$).

Thus one may build an affine 3-manifold using as a pattern a projective two manifold (open sets in the projective plane lift to open sets (cones) in V etc.). If we further divide by the action of a compactness is preserved in the construction.

The projective structures on the two torus constructed above yield compact affine 3-manifolds where the developing map is not a covering. In particular, from the example in Figure 7, we can obtain an affine 3-manifold which develops