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Inversive 2-manifolds
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# **INVERSIVE 2-MANIFOLDS**

Inversive structures on orientable two manifolds of genus >1 form a rich theory properly containing for example the classical subject of Fuchsian and Kleinian surface groups.

If  $Sl(2, \mathbb{C}) / \pm 1 = Gl(2, \mathbb{C})/Gl(1, \mathbb{C})$  is the group of fractional linear transformations of  $\mathbb{CP}^1$ , that is the group of orientable inversive (conformal) transformations of  $S^2$ , and  $\Gamma$  is a discrete subgroup acting freely and discontinuously on a connected open set  $\Omega \subset S^2$ , then  $\Omega/\Gamma$  is a 2-manifold M with inversive structure. M' is just  $\Omega$  and the developing map is an embedding.

*Example 1.* If  $\Gamma$  is a Fuchsian group, that is,  $\Omega$  is an open (round) disk in  $\mathbf{C} \subset S^2$ , then the inversive structure is actually a hyperbolic structure—corresponding to a metric of constant negative curvature. The structure is inversive and projective at the same time.

*Example 2.* If  $\Gamma$  as in Example 1 is deformed slightly (a so-called quasi-Fuchsian group; see [9]) then  $\Omega$  remains an open disk whose boundary can be a rather remarkable non rectifiable Jordan curve. This curve has no tangent at a dense set.



*Example 3.* Let  $\Gamma$  be generated by two general hyperbolic elements of sufficient strength so that the union of the fundamental domains of each covers the entire sphere. Then  $\Omega$  is  $S^2$  minus a Cantor set and  $\Omega/F$  is a compact conformal 2 manifold whose developing image is  $\Omega$ . (Shottky group) In Figure 3,  $r_1$ ,  $r_2$  and  $r_3$  are inversions (reflections) in three circles and  $\Gamma$  consists of all products of an even number of these inversions.  $\Gamma$  is generated by  $r_1r_2$  and  $r_1r_3$ . A fundamental domain is  $D \cup r_1D$ ,  $D = D_1 \cup D_2$ . The Cantor set appears clearly on the line of symmetry m.



FIGURE 3

*Example 4.* A class of examples not always arising from Kleinian groups as above can be achieved as follows. Let  $\gamma$  be the boundary of an immersed disk in  $S^2$ . Approximate  $\gamma$  by a closed immersed curve again bounding an immersed disk constituted of 2g + 2 (for some integer g > 0) successive arcs of circles meeting at right acute angles (Fig. 4). The new disk with scalloped edges has a conformal structure from the immersion and four of these may be assembled to obtain an inversive 2-manifold of genus g. This topological assemblage is suggested in Figure 5.



FIGURE 5

Note this construction uses inversion in circles, and four angles at a vertex add up to achieve the non singular conformal structure. Also note the original immersed disk may be chosen (for g big enough) to cover  $S^2$  completely (in a very complicated way) and then the developing map  $M' \rightarrow S^2$  cannot be a covering. In Figure 6 an example with immersed disk D with 6 vertices (g = 2) is suggested, where the developing map covers clearly  $S^2$  completely.



FIGURE 6

We note conversely that if the developing map  $M' \to S^2$  is not onto (see Fig. 3, where  $D_1$  is the initial disk, for an example) then the developing map is rather remarkably a covering of its image (Gunning [6]). The idea of the proof is the following—if the image omits at least three points, (exactly one or two points is easy) M' has a Poincaré metric of constant negative curvature preserved by the holonomy group of Moebius transformation acting on the image. Then the developing map becomes an isometric immersion of a complete manifold and thus a covering map.

*Example 5.* There are interesting projective structures on the torus constructed as follows. Start with a *generic* linear flow on the projective plane (with a source, a sink, and a saddle in point *B* in Fig. 7*a*) and choose an immersed curve transverse to the flow lines (Fig. 7*b*). Note that such curves may be based on a word in 2 symbols for example *ccaaaa* in Figure 7, and *ccaaacacaa* in Figure 8, where the closed curve on  $\mathbb{RP}^2$  is drawn on the open band that universally covers the Moebius band, projective plane minus point *B*.

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Flowing the curve along for time t sweeps out a thickening of the immersed curve, an immersed annulus. We may identify the two boundary components of the annulus by the time t map, a locally projective isomorphism.

The identification space is a projective structure on the torus M whose developing map is the map:  $M' = S^1 \times R \rightarrow \mathbf{RP}^2$ , obtained by spreading the immersed curve around by the flow for all time  $t \in \mathbf{R}$ .

The developing map is not a covering and the image is the projective plane minus three points for any word different from *aa* or *cc*. Note that the covering

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space M' is obtained by gluing, each time along one of the two segments of a or c, as many copies of open sectors bounded by the lines a and c, (each covering an open annulus [5]) as there are letters in the characteristic word. These projective structures on the 2-torus are characterized by their (cyclic) word and the t = 1 flow map. In suitable homogeneous coordinates the last is expressed as  $f_1: f_t: (x, y, z) \to (xe^{\alpha t}, ye^{\beta t}, ze^{\gamma t}) \alpha < \beta < \gamma, \quad t = 1.$ 

*Remark.* Following the curve from its initial point P to its endpoint P', one can say that the sectors of P and P' were identified by the identity map: in homogeneous coordinates.

$$(x, y, z) \rightarrow (x, y, z)$$

A more general case (see Goldman [5]) is obtained if we identify by any projectivity commuting with  $f_1$ :

$$g:(x, y, z) \rightarrow (xe^{\lambda}, ye^{\mu}, ze^{\nu})$$

 $\lambda, \mu, \nu \in \mathbf{R}$ .

## AFFINE STRUCTURES IN 2, 3, AND 4 DIMENSIONS

In dimension two only the torus admits an affine structure by Benzecri [1] and for all affine structures the developing map is a covering of its image by Nagano-Yagi [7]. The image is affinely equivalent to either the whole plane, the once punctured plane, the half plane or the quarter plane.

We obtain interesting affine structures in dimensions 3 and 4 using respectively the projective and inversive structures in dimension 2 discussed above.

i) A projective transformation of the real projective plane  $\mathbf{RP}^2 = \mathbf{R}^3 - \{0\}/\mathbf{R}^*$  (where  $\mathbf{R}^* = \mathbf{R} - \{0\}$ ) lifts to an affine transformation of  $V = \mathbf{R}^3 - \{0\}$ , unique but for scalar multiplication. Any such commutes with scalar multiplication by a real number  $\alpha > 1$  (e.g.  $\alpha = 2$ ).

Thus one may build an affine 3-manifold using as a pattern a projective two manifold (open sets in the projective plane lift to open sets (cones) in V etc.). If we further divide by the action of a compactness is preserved in the construction.

The projective structures on the two torus constructed above yield compact affine 3-manifolds where the developing map is not a covering. In particular, from the example in Figure 7, we can obtain an affine 3-manifold which develops