

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 29 (1983)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: SOME PARADOXICAL SETS WITH APPLICATIONS IN THE GEOMETRIC THEORY OF REAL VARIABLE
Autor: de Guzmán, Miguel
Kapitel: 8. The Nikodym set
DOI: <https://doi.org/10.5169/seals-52969>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 11.02.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

If we now perform an analogous construction starting from the other side BC of $ABCD$ we obtain a finite number of strips satisfying all the properties indicated (a), (b), (c).

The Besicovitch set is now very easily obtained as follows. We take a square $MNPQ$ of side length 1 and apply to it the above auxiliary construction (**) with $\varepsilon = 1/2$. We obtain a number of strips $\omega_1^1, \omega_2^1, \dots, \omega_{r_1}^1$, covering an area smaller than $1/2$ and such that for each segment determined by a point of MN and another of PQ there is a segment of the same length and direction inside $\Omega^1 = \omega_1^1 \cup \omega_2^1 \cup \dots \cup \omega_{r_1}^1$.

Now we consider each of the parallelograms ω_j^1 and apply to it the same construction (**) with $\varepsilon = 1/2^2 r_1$. Collecting all parallelograms corresponding to each $\omega_j^1, j = 1, 2, \dots, r_1$, we obtain a second family of parallelograms $\omega_1^2, \omega_2^2, \dots, \omega_{r_2}^2$. Their union $\Omega^2 = \omega_1^2 \cup \omega_2^2 \cup \dots \cup \omega_{r_2}^2$ has area less than $1/2$, is contained in Ω^1 and, again, for each segment joining a point of MN to another of PQ there is another one of the same length and direction inside Ω^2 . We proceed with the parallelograms ω_j^2 as we did with the ω_j^1 , now with $\varepsilon = 1/2^3 r_2$, and so on. Thus we obtain

$$\Omega^1 \supset \Omega^2 \supset \Omega^3 \supset \dots$$

of areas

$$S(\Omega^1) < 1/2, S(\Omega^2) < 1/2^2, S(\Omega^3) < 1/2^3, \dots$$

The sets Ω^j are compact and have the property of containing a parallel translation of each segment with one extremity on MN and the other on PQ . The intersection

$$B = \Omega^1 \cap \Omega^2 \cap \dots \cap \Omega^j \cap \dots$$

is of null area and has this same property. We now proceed with the square $MNPQ$ in the same way in the other direction and obtain a compact set of measure zero containing a segment of length one in each direction, i.e. the Besicovitch set.

8. THE NIKODYM SET

The Nikodym set can be obtained from the Perron tree in a similar way through the following auxiliary construction, also surprising in itself.

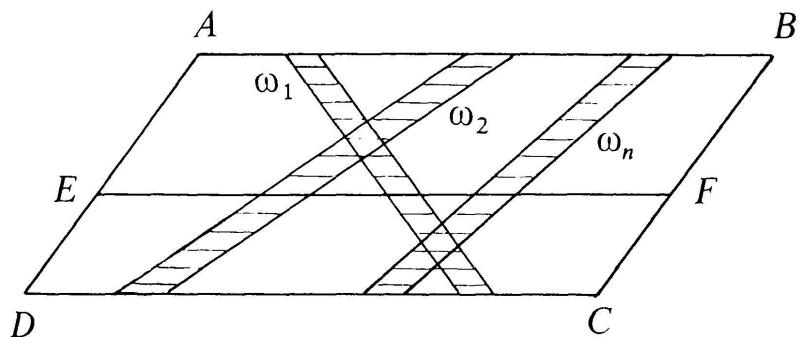


FIGURE 16

Let $ABCD$ be an arbitrary parallelogram and $CDEF$ another one contained in it as Figure 16 shows. Let ε be any arbitrary positive number. Then one can construct a finite number of parallelograms $\omega_1, \omega_2, \dots, \omega_q$, with a basis on CD and another one on AB such that the figure $\omega_1 \cup \omega_2 \cup \dots \cup \omega_q$ covers $CDEF$ while the part of it above EF has area less than ε , that is:

$$\omega_1 \cup \omega_2 \cup \dots \cup \omega_q \supset CDEF$$

$$S((\omega_1 \cup \omega_2 \cup \dots \cup \omega_q) \cap (ABCD - CDEF)) < \varepsilon$$

This construction is a little more technical than that of the Besicovitch set and will be omitted. For details we refer to Guzmán (1975).

9. MATHEMATICAL FRIVOLITIES?

FROM THE PERRON TREE TO THE MEASURE OF THE DENSITY

What started as a puzzle has proved to have many important applications to solve some interesting problems of recent analysis.

Let us assume that we have a mass distributed on the plane and that we wish to measure the density of this distribution at each point. Let us also suppose that the mass is not continuously distributed. One can perhaps say: "Will it not be very artificial to consider a mass that is not continuously distributed?" It is true that the old Scholastic used to affirm that "*natura non facit saltus*" (nature does not proceed by jumps). However, the findings of modern physics permit us to affirm with even stronger motivation "*natura non facit nisi saltus*" (nature proceeds only by jumps). Therefore it is rather natural to consider a discontinuous mass distribution.

For a long time one thought that in order to measure the density one could take *any system of reasonable sets* that contract to the point at which one measures the density, find the mean density over such sets and hope that, when