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Artikel: SOME PARADOXICAL SETS WITH APPLICATIONS IN THE GEOMETRIC THEORY OF REAL VARIABLE

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Kapitel: 6. The solution of the needle problem

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6. The solution of the needle problem

The Perron tree gives a simple solution to the Kakeya problem. First we shall show how a needle can go from a straight line to another one parallel to it covering an arbitrarily small area. Let us observe Figure 12.

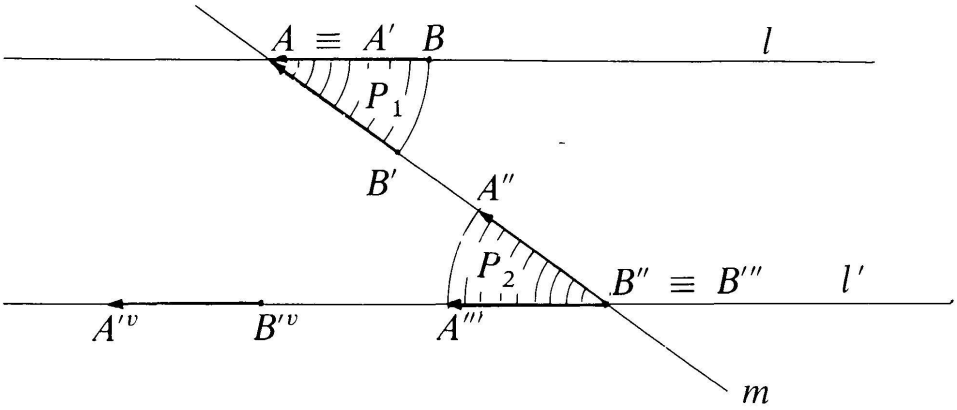

Figure 12

If the needle AB is on l and we wish to translate it to l^I, we draw through A a straight line m intersecting l and l^I whose direction can be as close to that of l and l^I as we wish. From AB we move to A^IB^I covering area P_1, from A^IB^I to $A^{II}B^{II}$ covering null area, from $A^{II}B^{II}$ to $A^{III}B^{III}$ covering P_2. Now $P_1 + P_2$ can be made arbitrarily small if the slope of m over l and l^I is small. From $A^{III}B^{III}$ we can move to any other position $A^{IV}B^{IV}$ on l^I covering again null area.

Let us now assume that the needle is on the side AB of the initial triangle ABC. We can assume that ABC is an equilateral triangle and that its height is of the same length as that of the needle. Let us see how we can move the needle to AC sweeping an area smaller than $\eta/3$ with a positive η arbitrarily small.

We construct a Perron tree P starting from ABC with an $\varepsilon > 0$ such that $\varepsilon S(ABC) < \eta/6$. Here, as before, $S(ABC)$ denotes the area of the triangle ABC. Let n be the number of small triangles $T_1, T_2, ..., T_n$ in which we have to divide ABC and let $T'_1 \equiv T_1, T'_2, ..., T'_n$ be their corresponding final positions in the Perron tree. We shall move the needle inside P and inside n figures like that of Figure 12 with an area J each one such that $nJ < \eta/6$. If the needle is on AB with an extremity on A, it can move inside $T'_1 \equiv T_1$, therefore inside P, until it comes over the right hand side of T'_1. Now T'_2 has its left hand side parallel to the right hand side of T'_1. Therefore it can move, using the above construction, covering an area J. Within T'_2, and so within P, it can move to the right hand side of T'_2. From there to the left hand side of T'_3 and so on until it comes to AC, covering area less than $\eta/3$.

It is clear that with three equilateral triangles and three repetitions of this process we can turn the needle around covering area smaller than η.

7. THE CONSTRUCTION OF THE BESOCOVITCH SET

The Besicovitch set is also easily built starting from the Perron tree by means of the following auxiliary construction:

(**) *Given an arbitrary parallelogram* $ABCD$ *and* $\varepsilon > 0$, *it is possible to construct a finite number of closed parallelograms* $\omega_1, \omega_2, ..., \omega_n$ *so that* (see Fig. 13):

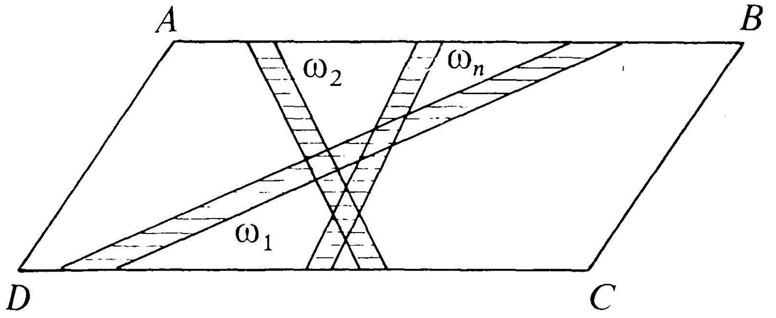

FIGURE 13

(a) *Each one has one basis on* AB *and another one on* CD.

(b) *The area of their union is less than* ε.

(c) *For each segment joining a point of* AB *to another one of* CD *there exists inside some* ω_j *a segment parallel to it of the same length.*

To see this, given $ABCD$ and $\varepsilon > 0$ we first take two strips ω_1 and ω_2 as indicated in Figure 14 such that $S(\omega_1) + S(\omega_2) < \varepsilon/4$. We take now a point L of UV so that LC is parallel to UT. Then we divide VC into intervals with the same length smaller than that of DV and we join L to the extreme points of these intervals. A typical triangle of the ones so obtained is LMN. Let p be