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Artikel: SOME PARADOXICAL SETS WITH APPLICATIONS IN THE GEOMETRIC THEORY OF REAL VARIABLE

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Kapitel: 4. A SMALL TREE WITH MANY FRUITS

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3. A TOURIST COLONY NOT TO BE RECOMMENDED

In 1927 Nikodym, in order to explore the geometric structure of the measurable sets in the plane, showed how to construct, inside a square Q, a set N that fills it (i.e. the measure of Q-N is zero) and so that for each point x of N there is a straight line $l(x)$ passing through it and not hitting any other point of N (Fig. 4).

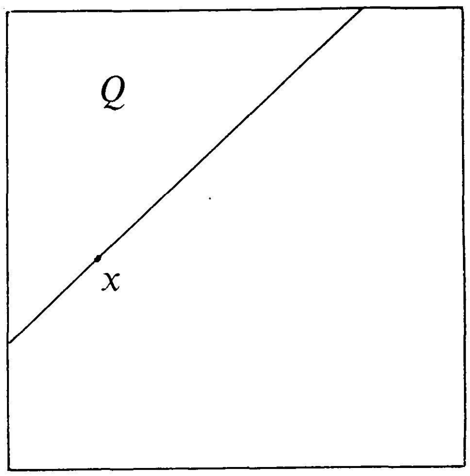

FIGURE 4

The architects of tourist colonies have not yet learned about this magnificent business possibility, but the day somebody tells you about the marvels of a colony in an island which offers a free view over the ocean from each one of its apartments, beware!

Although it seems incredible one can still make it better. R. O. Davies in 1953 constructed a set N in Q filling Q and such that each x of N has infinitely many directions in which one can see the ocean... inside any arbitrarily small angle one may fix!

4. A SMALL TREE WITH MANY FRUITS

In 1928 Besicovitch was informed about the needle problem and published its solution. In 1929 Perron simplified the somewhat laborious construction of Besicovitch. It has been further simplified later on. The final product of the line of thought, that we shall call *the Perron tree*, has proved to be an extraordinarily fruitful tool for the solution of certain deep problems of recent mathematical analysis.

The result is as follows: Given an arbitrary $\varepsilon > 0$ and an arbitrary triangle ABC of area that we denote by $S(ABC)$, we can divide the triangle ABC into small triangles $T_1, T_2, \ldots, T_n$ as Figure 5 shows (i.e. dividing the basis a into a finite number of equal intervals $I_1, I_2, \ldots, I_n$) and one can translate appropriately the small triangles $T_1, T_2, \ldots, T_n$ parallelly to the basis a in such a way that the area of the union of the translated triangles is less than $\varepsilon S(ABC)$. (See Fig. 6.)

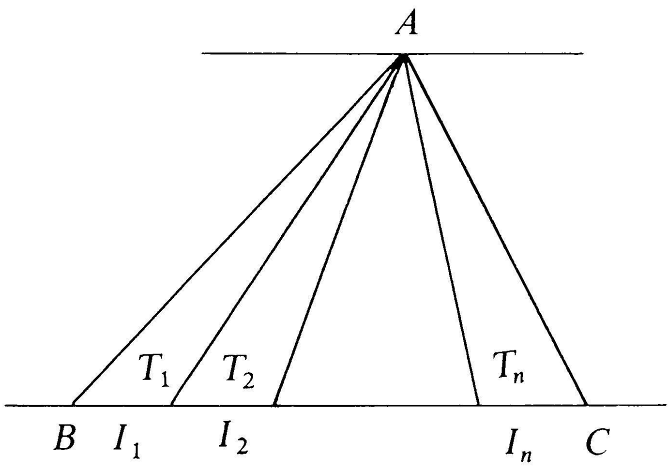

FIGURE 5

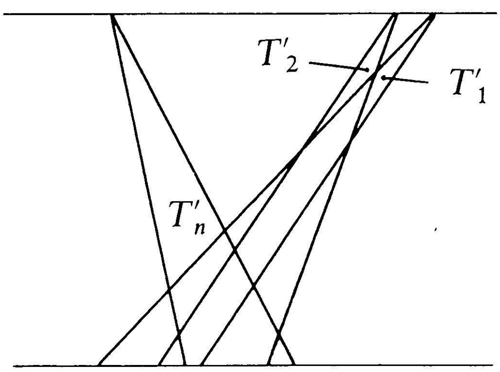

FIGURE 6

5. HOW THE PERRON TREE SPROUTS

Following an idea of Rademacher (1962), the construction of the Perron tree can be easily understood as follows. Let us divide first a triangle T, MNP, of area $S(T)$, into two triangles T_1, T_2, with bases J_1, J_2, of the same length. If we wish to move T_1 and T_2, parallelly to NP so that the shifted triangles cover less area we can do it by pushing T_2 towards T_1 as Figure 7 shows. The area covered by T_1 and T'_2 can be easily measured by elementary geometry and is (see Fig. 7, we take $1/2 < \alpha < 1$)

$$\alpha^2 S(T) + 2(1-\alpha)^2 S(T)$$