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$\mathcal{A}^0 \cap \mathcal{B}$ is spanned by the elements (4.2) such that $b = c$, all of which are of the form $J^a E_+^b$. \square

We remark that the subalgebra of \mathcal{A} generated by \mathfrak{sl}_2 is canonically isomorphic to the universal enveloping algebra of \mathfrak{sl}_2 . The element $J(J+2)$ equals $H^2 + 2(E_+E_- + E_-E_+)$, the Casimir element for \mathfrak{sl}_2 . Thus \mathcal{A}^0 is a little larger than the enveloping algebra of \mathfrak{sl}_2 .

For integers l, n define $\mathcal{B} \binom{n}{l}$ to be the set of $T \in \mathcal{B} \cap \mathcal{A}^n$ such that $\rho(H)T = lT$.

This defines a grading of \mathcal{B} :

$$\mathcal{B} = \bigoplus \mathcal{B} \binom{n}{l}, \quad \mathcal{B} \binom{n}{l} \cdot \mathcal{B} \binom{n'}{l'} \subset \mathcal{B} \binom{n+n'}{l+l'}. \quad (4.8)$$

The generators of \mathcal{B} fit in as follows:

$$J \in \mathcal{B} \binom{0}{0}, \quad X \in \mathcal{B} \binom{1}{1}, \quad \partial_Y \in \mathcal{B} \binom{-1}{1}. \quad (4.9)$$

PROPOSITION 4.10. i) $\mathcal{B} \binom{0}{0} = \mathbf{C}[J]$.

ii) $\mathcal{B} \binom{n}{l} \neq 0$ if and only if $l \geq 0, |n| \leq l$, and $l \equiv n \pmod{2}$. If these conditions are met, then

$$\mathcal{B} \binom{n}{l} = \mathbf{C}[J] \cdot X^{\frac{l+n}{2}} (\partial_Y)^{\frac{l-n}{2}} \quad (4.11)$$

Proof: Immediate. \square

We note that the condition that $\mathcal{B} \binom{n}{l} \neq (0)$ may be rephrased thus: $l \geq 0$ and n is a weight of V_l .

5. DECOMPOSITION OF $\text{Hom}(V_m, V_{m+n})$

THEOREM 5.1. Let l, m, n be integers with $l, m, m+n \geq 0$. There is an \mathfrak{sl}_2 -subrepresentation of $\text{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ which is isomorphic to V_l if and only if $|n| \leq l, n \equiv l \pmod{2}$, and $m \geq \frac{l-n}{2}$.

Moreover, when these conditions are met there is a unique such subrepresentation. A weight vector of weight l in it is given by

$$X^{\frac{l+n}{2}}(\partial_Y)^{\frac{l-n}{2}}.$$

Proof: By Lemma 2.7 and the definition of $\mathcal{B}\left(\begin{smallmatrix} n \\ l \end{smallmatrix}\right)$, a weight vector of weight l of the subrepresentation sought must be the restriction to V_m of an element of $\mathcal{B}\left(\begin{smallmatrix} n \\ l \end{smallmatrix}\right)$. By Lemma 4.10ii, all such restrictions are scalar multiples of the restriction of $X^{\frac{l+n}{2}}(\partial_Y)^{\frac{l-n}{2}}$ to V_m , which restriction is nonzero only when $m \geq \frac{l-n}{2}$. □

It is interesting to observe that the weight l weight vector in $\text{Hom}_{\mathbb{C}}(V_m, V_{m+n})$ given by Theorem 5.1 is “independent” of m .

Finally we want to give formulas for the weight vectors in $\text{Hom}(V_m, V_{m+n})$ of all weights, not just of highest weight.

For integers l, i, j with $l \geq 0$ and $0 \leq i, j \leq l$, define an element $A_l(i, j)$ of \mathcal{A} :

$$A_l(i, j) = \sum_{\alpha \leq k \leq \beta} (-1)^k \binom{l}{i} \binom{i}{k} \binom{l-i}{j-k} X^{l-i-j+k} Y^{j-k} (\partial_X)^k (\partial_Y)^{i-k}$$

with $\alpha = \sup\{0, i+j-l\}$ and $\beta = \inf\{i, j\}$. (5.2)

LEMMA 5.3. $\rho(E_-)^j \binom{l}{i} X^{l-i} (\partial_Y)^i = j! A_l(i, j).$

Proof: By induction on j . Use the formula:

$$[E_-, D(i, j, a, b)] = iD(i-1, j+1, a, b) - bD(i, j, a+1, b-1)$$

with D as in (2.1). □

THEOREM 5.4. Let l, m, n be such that there is a subrepresentation of $\text{Hom}_{\mathbb{C}}(V_m, V_{m+n})$ isomorphic to V_l . Then an inclusion of representations $\phi: V_l \rightarrow \text{Hom}_{\mathbb{C}}(V_m, V_{m+n})$ may be given by the formula:

$$\phi(X^{l-j} Y^j) = \frac{1}{\binom{l}{j}} A_l\left(\frac{l-n}{2}, j\right). \tag{5.5}$$

Proof: This depends on (5.3) and the calculation in V_l that

$$E_{-j}X^l = \frac{l!}{(l-j)!} X^{l-j}Y^j. \quad \square$$

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