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$\mathscr{A}^0 \cap \mathscr{B}$ is spanned by the elements (4.2) such that $b = c$, all of which are of the form $J^a E_+{}^b$. □

We remark that the subalgebra of $\mathscr{A}$ generated by $\mathfrak{sl}_2$ is canonically isomorphic to the universal enveloping algebra of $\mathfrak{sl}_2$. The element $J(J+2)$ equals $H^2 + 2(E_+E_- + E_-E_+)$, the Casimir element for $\mathfrak{sl}_2$. Thus $\mathscr{A}^0$ is a little larger than the enveloping algebra of $\mathfrak{sl}_2$.

For integers l, n define $\mathscr{B}\binom{n}{l}$ to be the set of $T \in \mathscr{B} \cap \mathscr{A}^n$ such that $\rho(H)T = lT$.

This defines a grading of $\mathscr{B}$:

$$\mathscr{B} = \oplus\, \mathscr{B}\binom{n}{l}, \qquad \mathscr{B}\binom{n}{l} \cdot \mathscr{B}\binom{n'}{l'} \subset \mathscr{B}\binom{n+n'}{l+l'}. \tag{4.8}$$

The generators of $\mathscr{B}$ fit in as follows:

$$J \in \mathscr{B}\binom{0}{0}, \qquad X \in \mathscr{B}\binom{1}{1} \qquad \partial_Y \in \mathscr{B}\binom{-1}{1}. \tag{4.9}$$

PROPOSITION 4.10. *i)* $\mathscr{B}\binom{0}{0} = \mathbf{C}[J]$.

ii) $\mathscr{B}\binom{n}{l} \neq 0$ *if and only if* $l \geqslant 0, |\, n \,| \leqslant l$, *and* $l \equiv n \,(\mathrm{mod}\ 2)$. *If these conditions are met, then*

$$\mathscr{B}\binom{n}{l} = \mathbf{C}[J] \cdot X^{\frac{l+n}{2}} (\partial_Y)^{\frac{l-n}{2}} \tag{4.11}$$

Proof: Immediate. □

We note that the condition that $\mathscr{B}\binom{n}{l} \neq (0)$ may be rephrased thus: $l \geqslant 0$ and n is a weight of V_l.

5. DECOMPOSITION OF $\mathrm{Hom}(V_m, V_{m+n})$

THEOREM 5.1. *Let* l, m, n *be integers with* $l, m, m+n \geqslant 0$. *There is an* $\mathfrak{sl}_2$*-subrepresentation of* $\mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ *which is isomorphic to* V_l *if and only if* $|\, n \,| \leqslant l, n \equiv l \,(\mathrm{mod}\ 2)$, *and* $m \geqslant \dfrac{l-n}{2}$.

Moreover, when these conditions are met there is a unique such subrepresentation. A weight vector of weight l *in it is given by*

$$X^{\frac{l+n}{2}}(\partial_Y)^{\frac{l-n}{2}}.$$

Proof: By Lemma 2.7 and the definition of $\mathscr{B}\binom{n}{l}$, a weight vector of weight l of the subrepresentation sought must be the restriction to V_m of an element of $\mathscr{B}\binom{n}{l}$. By Lemma 4.10ii, all such restrictions are scalar multiples of the restriction of $X^{\frac{l+n}{2}}(\partial_Y)^{\frac{l-n}{2}}$ to V_m, which restriction is nonzero only when $m \geqslant \frac{l-n}{2}$. □

It is interesting to observe that the weight l weight vector in $\mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ given by Theorem 5.1 is "independent" of m.

Finally we want to give formulas for the weight vectors in $\mathrm{Hom}(V_m, V_{m+n})$ of all weights, not just of highest weight.

For integers l, i, j with $l \geqslant 0$ and $0 \leqslant i,j \leqslant l$, define an element $A_l(i,j)$ of $\mathscr{A}$:

$$A_l(i,j) = \sum_{\alpha \leqslant k \leqslant \beta} (-1)^k \binom{l}{i}\binom{i}{k}\binom{l-i}{j-k} X^{l-i-j+k}\, Y^{j-k}(\partial_X)^k(\partial_Y)^{i-k}$$

$$\text{with } \alpha = \sup\{0, i+j-l\} \quad \text{and} \quad \beta = \inf\{i,j\}. \tag{5.2}$$

LEMMA 5.3. $\rho(E_-)^j \binom{l}{i} X^{l-i}(\partial_Y)^i = j!\, A_l(i,j)$.

Proof: By induction on j. Use the formula:

$$[E_-, D(i,j,a,b)] = iD(i-1, j+1, a, b) - bD(i,j,a+1,b-1)$$

with D as in (2.1). □

THEOREM 5.4. *Let* l, m, n *be such that there is a subrepresentation of* $\mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ *isomorphic to* V_l. *Then an inclusion of representations* $\phi: V_l \to \mathrm{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ *may be given by the formula:*

$$\phi(X^{l-j}Y^j) = \frac{1}{\binom{l}{j}} A_l\left(\frac{l-n}{2}, j\right). \tag{5.5}$$

Proof: This depends on (5.3) and the calculation in V_l that

$$E_-^{\,j} X^l = \frac{l!}{(l-j)!} X^{l-j} Y^j . \qquad \square$$

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