

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 29 (1983)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE CLEBSCH-GORDAN FORMULAS
Autor: Flath, Daniel
Kapitel: 5. Decomposition of $\text{Hom}(V_m, V_{\{m+n\}})$
DOI: <https://doi.org/10.5169/seals-52986>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 19.08.2025

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

$\mathcal{A}^0 \cap \mathcal{B}$ is spanned by the elements (4.2) such that $b = c$, all of which are of the form $J^a E_+^b$. \square

We remark that the subalgebra of \mathcal{A} generated by \mathfrak{sl}_2 is canonically isomorphic to the universal enveloping algebra of \mathfrak{sl}_2 . The element $J(J+2)$ equals $H^2 + 2(E_+E_- + E_-E_+)$, the Casimir element for \mathfrak{sl}_2 . Thus \mathcal{A}^0 is a little larger than the enveloping algebra of \mathfrak{sl}_2 .

For integers l, n define $\mathcal{B} \binom{n}{l}$ to be the set of $T \in \mathcal{B} \cap \mathcal{A}^n$ such that $\rho(H)T = lT$.

This defines a grading of \mathcal{B} :

$$\mathcal{B} = \bigoplus \mathcal{B} \binom{n}{l}, \quad \mathcal{B} \binom{n}{l} \cdot \mathcal{B} \binom{n'}{l'} \subset \mathcal{B} \binom{n+n'}{l+l'}. \quad (4.8)$$

The generators of \mathcal{B} fit in as follows:

$$J \in \mathcal{B} \binom{0}{0}, \quad X \in \mathcal{B} \binom{1}{1}, \quad \partial_Y \in \mathcal{B} \binom{-1}{1}. \quad (4.9)$$

PROPOSITION 4.10. i) $\mathcal{B} \binom{0}{0} = \mathbf{C}[J]$.

ii) $\mathcal{B} \binom{n}{l} \neq 0$ if and only if $l \geq 0, |n| \leq l$, and $l \equiv n \pmod{2}$. If these conditions are met, then

$$\mathcal{B} \binom{n}{l} = \mathbf{C}[J] \cdot X^{\frac{l+n}{2}} (\partial_Y)^{\frac{l-n}{2}} \quad (4.11)$$

Proof: Immediate. \square

We note that the condition that $\mathcal{B} \binom{n}{l} \neq (0)$ may be rephrased thus: $l \geq 0$ and n is a weight of V_l .

5. DECOMPOSITION OF $\text{Hom}(V_m, V_{m+n})$

THEOREM 5.1. Let l, m, n be integers with $l, m, m+n \geq 0$. There is an \mathfrak{sl}_2 -subrepresentation of $\text{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ which is isomorphic to V_l if and only if $|n| \leq l, n \equiv l \pmod{2}$, and $m \geq \frac{l-n}{2}$.

Moreover, when these conditions are met there is a unique such subrepresentation. A weight vector of weight l in it is given by

$$X^{\frac{l+n}{2}}(\partial_Y)^{\frac{l-n}{2}}.$$

Proof: By Lemma 2.7 and the definition of $\mathcal{B} \binom{n}{l}$, a weight vector of weight l of the subrepresentation sought must be the restriction to V_m of an element of $\mathcal{B} \binom{n}{l}$. By Lemma 4.10ii, all such restrictions are scalar multiples of the restriction of $X^{\frac{l+n}{2}}(\partial_Y)^{\frac{l-n}{2}}$ to V_m , which restriction is nonzero only when $m \geq \frac{l-n}{2}$. \square

It is interesting to observe that the weight l weight vector in $\text{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ given by Theorem 5.1 is “independent” of m .

Finally we want to give formulas for the weight vectors in $\text{Hom}(V_m, V_{m+n})$ of all weights, not just of highest weight.

For integers l, i, j with $l \geq 0$ and $0 \leq i, j \leq l$, define an element $A_l(i, j)$ of \mathcal{A} :

$$A_l(i, j) = \sum_{\alpha \leq k \leq \beta} (-1)^k \binom{l}{i} \binom{i}{k} \binom{l-i}{j-k} X^{l-i-j+k} Y^{j-k} (\partial_X)^k (\partial_Y)^{i-k}$$

with $\alpha = \sup\{0, i+j-l\}$ and $\beta = \inf\{i, j\}$. (5.2)

LEMMA 5.3. $\rho(E_-)^j \binom{l}{i} X^{l-i} (\partial_Y)^i = j! A_l(i, j)$.

Proof: By induction on j . Use the formula:

$$[E_-, D(i, j, a, b)] = iD(i-1, j+1, a, b) - bD(i, j, a+1, b-1)$$

with D as in (2.1). \square

THEOREM 5.4. Let l, m, n be such that there is a subrepresentation of $\text{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ isomorphic to V_l . Then an inclusion of representations $\phi: V_l \rightarrow \text{Hom}_{\mathbf{C}}(V_m, V_{m+n})$ may be given by the formula:

$$\phi(X^{l-j} Y^j) = \frac{1}{\binom{l}{j}} A_l \left(\frac{l-n}{2}, j \right). \quad (5.5)$$

Proof: This depends on (5.3) and the calculation in V_l that

$$E_-^j X^l = \frac{l!}{(l-j)!} X^{l-j} Y^j.$$

□

REFERENCES

- [1] L. C. BIEDENHARN and J. D. LOUCK. *Angular Momentum in Quantum Physics*. Addison-Wesley (Reading, Massachusetts), 1981.
- [2] D. FLATH and L. C. BIEDENHARN. Beyond the Enveloping Algebra of \mathfrak{sl}_3 . *Preprint*.
- [3] A. A. KIRILLOV. *Elements of the Theory of Representations*. Springer-Verlag (Berlin), 1976.

(Reçu le 9 mai 1983)

Daniel Flath

Department of Mathematics
 Duke University
 Durham, North Carolina 27706
 USA